M233 Fall 2005 Homework 1
Solutions

1. Given $\mathbf{u} = \langle 5, -2, 3\rangle$ and $\mathbf{v} = \langle 2, -1, 2\rangle$, find nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ such that (i) $\mathbf{a} \parallel \mathbf{v}$, (ii) $\mathbf{a} \perp \mathbf{b}$, and (iii) $\mathbf{u} = \mathbf{a} + \mathbf{b}$.

Solution For the vector $\mathbf{a}$ we set

$$\mathbf{a} = \text{proj}_{\mathbf{v}}(\mathbf{u}) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = \left(\frac{18}{9}\right)\langle 2, -1, 2\rangle = \langle 4, -2, 4\rangle .$$

By construction we have $\mathbf{a} \parallel \mathbf{v}$. Next, set

$$\mathbf{b} = \mathbf{u} - \mathbf{a} = \langle 5, -2, 3\rangle - \langle 4, -2, 4\rangle = \langle 1, 0, -1\rangle$$

(from which the equation $\mathbf{u} = \mathbf{a} + \mathbf{b}$ is immediate). Finally, $\mathbf{a}\cdot\mathbf{b} = (4)(1) + (-2)(0) + (4)(-1) = 0$, from which the last required property, $\mathbf{a} \perp \mathbf{b}$, follows.

2. Let $\mathbf{u} = \langle 1, -1, 2\rangle$, $\mathbf{v} = \langle 2, 1, 1\rangle$, and $\mathbf{w} = \langle 3, 4, -2\rangle$. Find scalars s and t such that $\mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = s\mathbf{u} + t\mathbf{v}$.

Solution: According to a formula for the triple vector product, we have

$$\mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{w}\cdot\mathbf{v})\,\mathbf{u} - (\mathbf{w}\cdot\mathbf{u})\,\mathbf{v}.$$

We calculate $s = \mathbf{w}\cdot\mathbf{v} = 8$ and $t = \mathbf{w}\cdot\mathbf{u} = -5$. As a check,

$$(\mathbf{w}\cdot\mathbf{v})\,\mathbf{u} - (\mathbf{w}\cdot\mathbf{u})\,\mathbf{v} = 8\langle 1, -1, 2\rangle - (-5)\langle 2, 1, 1\rangle = \langle 18, -3, 21\rangle$$

and

$$\mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = \langle 3, 4, -2\rangle \times \langle -3, 3, 3\rangle = \langle 18, -3, 21\rangle .$$

3. A plane has parametric equations $x = 1 + 2s + t$, $y = 2 - s + 3t$, $z = 4 + s + t$. A line has symmetric equations $x/2 = y + 1 = 7(z - 1)$. Find their point of intersection.

Solution: In vector form, the equation of the plane is

$$\langle x, y, z\rangle = \langle 1, 2, 4\rangle + s\langle 2, -1, 1\rangle + t\langle 1, 3, 1\rangle .$$

A normal to the plane is therefore $\langle 2,-1,1\rangle \times \langle 1,3,1\rangle = \langle -4,-1,7\rangle$. Therefore, the plane has Cartesian equation $-4\,(x-1)-1\,(y-2)+7\,(z-4)=0$. The line has parametric equations $x=2t$, $y=t-1$, $z=t/7+1$. We substitute these values for x, y, and z into the equation of the plane, obtaining $-4\,(2t-1)-((t-1)-2)+7\,(t/7+1-4)=0$. We solve this equation to find $t=-7/4$. The corresponding point is

$$(x,y,z)=(2t,t-1,t/7+1)=(-7/2,-11/4,3/4)\,.$$

4. Find symmetric equations for the line of intersection of the two planes $x+2y-z=1$ and $2x+y+2z=2$.

Solution: The line of intersection of the two planes is parallel to $\langle 1,2,-1\rangle \times \langle 2,1,2\rangle$, which evaluates to $\langle 5,-4,-3\rangle$. We find a point on the line by setting $y=0$ (an arbitrary value chosen for its simplicity) and solve the two equations, $x+0-z=1$ and $2x+0+2z=2$, simultaneously for x and z. The result is $x=1$, $z=0$. Therefore, the point $(1,0,0)$ lies on the line of intersection. Symmetric equations for the line are

$$\frac{x-1}{5}=\frac{y}{-4}=\frac{z}{-3}.$$

5. A plane P intersects the coordinate axes in the points $(2,0,0)$, $(0,3,0)$, and $(0,0,5)$. A line ℓ that is perpendicular to P passes through the point $(-1,2,2)$. At what point does ℓ intersect the xy-plane?

Solution: In analogy with the intercept form of the equation of a line in the xy-plane, we quickly see that the three given points satisfy the equation

$$\frac{x}{2}+\frac{y}{3}+\frac{z}{5}=1,$$

which therefore is the Cartesian equation of P. It follows that line ℓ has vector equation

$$\langle x,y,z\rangle = \langle -1,2,2\rangle + t\left\langle \frac{1}{2},\frac{1}{3},\frac{1}{5}\right\rangle$$

and parametric equations $x=-1+t/2$, $y=2+t/3$, $z=2+t/5$. We see that $z=0$ for $t=-10$. For this value of t we have $x=-1+(-10)/2=-6$ and $y=2+(-10)/3=-4/3$. Therefore, the point of intersection is $(-6,-4/3,0)$.