SOLUTIONS TO EXERCISES FOR SECTION 7.5

Problems for Practice

In each of Exercises 1-10, write down the *form* of the partial fractions decomposition of the given rational function. *Do not explicitly calculate the coefficients.*

1. Since x^2+1 and x^2+4 are irreducible,

$$\frac{2x^3+x+1}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}$$

2. Since x^2+2x+2 is irreducible and $2x^2+5x+3 = (2x+3)(x+1)$, we have

$$\frac{2x^4}{(x^2+2x+2)(2x^2+5x+3)} = \frac{2x^4}{2x^4+9x^3+17x^2+16x+6}$$

$$= 1 + \frac{\left(-9x^3-17x^2-16x-6\right)}{(x^2+2x+2)(2x+3)(x+1)} = 1 + \frac{Ax+B}{x^2+2x+2} + \frac{C}{2x+3} + \frac{D}{x+1}.$$

3. Since x^2+1 and x^2+x+1 are irreducible, we have

$$\frac{2x^3+x+1}{(x^2+1)(x^2+x+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{(x^2+x+1)^2}.$$

4. Since x^2+1 and $17x^2+4x+1$ are irreducible, and since

$$\frac{2x^4}{(x^2+1)(17x^2+4x+1)} = \frac{2}{17} + \frac{r(x)}{(x^2+1)(17x^2+4x+1)}$$

where $r(x)$ is a degree 3 polynomial, we have

$$\frac{2x^4}{(x^2+1)(17x^2+4x+1)} = \frac{2}{17} + \frac{Ax+B}{x^2+1} + \frac{Cx+D}{17x^2+4x+1}.$$

5. Since x^2+x+3 is irreducible, we have

$$\frac{2x+1}{(x^2+x+3)(x-4)} = \frac{Ax+B}{x^2+x+3} + \frac{C}{x-4}.$$

6. Since x^2 is a *reducible* quadratic, we have

$$\frac{3x^3+x+1}{x^2(x+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}.$$

7. Since x^2+4 is irreducible, we have

$$\frac{2x^6}{(x^2+4)^3(x-2)} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2} + \frac{Ex+F}{(x^2+4)^3} + \frac{G}{x-2}.$$

8. We have

$$\frac{3x^5+x+1}{x^2(x^2-1)^2} = \frac{3x^5+x+1}{x^2(x-1)^2(x+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{x+1} + \frac{F}{(x+1)^2}.$$

9. Since x^2+1 is irreducible but $x^2 = x \cdot x$ and $x^2-1 = (x-1)(x+1)$ are not, we have

$$\frac{3x^5+x+1}{x^2(x^2-1)(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{x+1} + \frac{Ex+F}{x^2+1}.$$

10. Since $x^2+6x+10$ is irreducible, we have

$$\frac{7x^8}{(x-3)^3(x^2+6x+10)^3} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{(x-3)^3} + \frac{Dx+E}{x^2+6x+10} + \frac{Fx+G}{(x^2+6x+10)^2} + \frac{Hx+I}{(x^2+6x+10)^3}.$$

In each of Exercises 11-20, explicitly calculate the partial fractions decomposition of the given rational function.

11. We set

$$\frac{3x^2-5x+4}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} = \frac{A\left(x^2+1\right)+(Bx+C)(x-1)}{(x-1)(x^2+1)}.$$

Equating numerators, we obtain $3x^2-5x+4 = A\left(x^2+1\right)+(Bx+C)(x-1)$. When $x=1$ this equation becomes $3-5+4 = A\left(1^2+1\right)$, or $A=1$. Substituting this value in the equation of the numerators results in $3x^2-5x+4 = \left(x^2+1\right)+(Bx+C)(x-1)$, or $2x^2-5x+3 = (Bx+C)(x-1)$, or $(2x-3)(x-1) = (Bx+C)(x-1)$. It follows that $2x-3 = Bx+C$, or $B=2$ and $C=-3$. Thus,

$$\frac{3x^2-5x+4}{(x-1)(x^2+1)} = \frac{1}{x-1} + \frac{2x-3}{x^2+1}.$$

12. We set

$$\frac{x^2+2}{(x^2+1)x^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x} + \frac{D}{x^2} = \frac{(Ax+B)x^2+Cx\left(x^2+1\right)+D\left(x^2+1\right)}{(x^2+1)x^2}.$$

Equating numerators, we obtain $x^2+2 = (Ax+B)x^2+Cx\left(x^2+1\right)+D\left(x^2+1\right)$. When $x=0$ this equation becomes $2=D$. Substituting this value into the equation of the numerators results in $x^2+2 = (Ax+B)x^2 + Cx\left(x^2+1\right)+2\left(x^2+1\right)$, or $-x^2 = (Ax+B)x^2+Cx\left(x^2+1\right)$, or $0x^3+(-1)x^2+0x = (A+C)x^3+Bx^2+Cx$. Equating the coefficients of like powers of x, we obtain $A+C=0$, $B=-1$, and $C=0$. It follows that $A=C=0$ and

$$\frac{x^2+2}{(x^2+1)x^2} = \frac{-1}{x^2+1} + \frac{2}{x^2}.$$

13. We set

$$\frac{7x^3+9x-3x^2-6}{(x^2+2)(x^2+1)} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{x^2+1} = \frac{(Ax+B)\left(x^2+1\right)+(Cx+D)\left(x^2+2\right)}{(x^2+2)(x^2+1)}.$$

Equating numerators, we obtain $7x^3-3x^2+9x-6 = (Ax+B)\left(x^2+1\right)+(Cx+D)\left(x^2+2\right) = (A+C)x^3 + (B+D)x^2+(A+2C)x+(B+2D)$. Equating the coefficients of like powers of x, we obtain $A+C=7$, $B+D=-3$, $A+2C=9$, and $B+2D=-6$. We solve the pair of equations in A and C simultaneously and do the same for B and D. It follows that $A=5$, $C=2$, $B=0$, and $D=-3$. Thus,

$$\frac{7x^3-3x^2+9x-6}{(x^2+2)(x^2+1)} = \frac{5x}{x^2+2} + \frac{2x-3}{x^2+1}.$$

14. We set

$$\frac{x^2+2x}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} = \frac{\left(x^2+1\right)(Ax+B)+Cx+D}{(x^2+1)^2}.$$

Equating numerators, we obtain $x^2+2x = \left(x^2+1\right)(Ax+B)+Cx+D$, or $x^2+2x = Ax^3+Bx^2+(A+C)x+(B+D)$. Equating the coefficients of like powers of x yields the equations $A=0$, $B=1$, $A+C=2$, and $B+D=0$. It follows that $C=2$, $D=-1$, and

$$\frac{x^2+2x}{(x^2+1)^2} = \frac{1}{x^2+1} + \frac{2x-1}{(x^2+1)^2}.$$

15. We set

$$\frac{x^3-x}{(x^2+1)^2}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{\left(x^2+1\right)^2}=\frac{\left(x^2+1\right)(Ax+B)+Cx+D}{\left(x^2+1\right)^2}.$$

Equating numerators, we obtain $x^3-x=\left(x^2+1\right)(Ax+B)+Cx+D$, or $x^3-x=Ax^3+Bx^2+(A+C)\,x+(B+D)$. Equating the coefficients of like powers of x yields the equations $A=1$, $B=0$, $A+C=-1$, and $B+D=0$. It follows that $C=-2$, $D=0$, and

$$\frac{x^3-x}{\left(x^2+1\right)^2}=\frac{x}{x^2+1}-\frac{2x}{\left(x^2+1\right)^2}.$$

16. We set

$$\frac{2x^2}{(x+1)^2\left(x^2+1\right)}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{Cx+D}{x^2+1}=\frac{A\,(x+1)\left(x^2+1\right)+B\left(x^2+1\right)+(Cx+D)\,(x+1)^2}{(x+1)^2\left(x^2+1\right)}.$$

Equating numerators, we obtain $2x^2=A\,(x+1)\left(x^2+1\right)+B\left(x^2+1\right)+(Cx+D)\,(x+1)^2$. For $x=-1$ this equation results in $2=B\left((-1)^2+1\right)$, or $B=1$. Substituting this value for B into the equation of the numerators gives us $2x^2=A\,(x+1)\left(x^2+1\right)+\left(x^2+1\right)+(Cx+D)\,(x+1)^2$, or $x^2-1=A\,(x+1)\left(x^2+1\right)+(Cx+D)\,(x+1)^2$. Dividing each side by $x+1$ yields $x-1=A\left(x^2+1\right)+(Cx+D)\,(x+1)$, or $x-1=(A+C)\,x^2+(C+D)\,x+(A+D)$. Equating the coefficients of like powers of x yields the equations $A+C=0$, $C+D=1$, and $A+D=-1$. Since $A=-C$ (from the first of these three equations), the last equation becomes $-C+D=-1$. Solving this equation simultaneously with the middle equation $C+D=1$ results in $C=1$ and $D=0$. It follows that $A=-1$ and

$$\frac{2x^2}{(x+1)^2\left(x^2+1\right)}=\frac{-1}{x+1}+\frac{1}{(x+1)^2}+\frac{x}{x^2+1}.$$

17. We set

$$\frac{x^3+12x^2-9x+48}{(x-3)\left(x^2+4\right)}=1+\frac{15x^2-13x+60}{(x-3)\left(x^2+4\right)}=1+\frac{A}{x-3}+\frac{Bx+C}{x^2+4}=\frac{A\left(x^2+4\right)+(Bx+C)\,(x-3)}{(x-3)\left(x^2+4\right)}.$$

Equating numerators yields $15x^2-13x+60=A\left(x^2+4\right)+(Bx+C)\,(x-3)$. For $x=3$ this equation becomes $15\,(9)-13\,(3)+60=A\left(3^2+4\right)$, or $A=12$. Replacing A with 12 in the equation of the numerators results in $15x^2-13x+60=12\left(x^2+4\right)+(Bx+C)\,(x-3)$, or $3x^2-13x+12=(Bx+C)\,(x-3)$. Factoring the left side gives us $(3x-4)\,(x-3)=(Bx+C)\,(x-3)$, from which we deduce $B=3$ and $C=-4$. Thus

$$\frac{x^3+12x^2-9x+48}{(x-3)\left(x^2+4\right)}=1+\frac{12}{x-3}+\frac{3x-4}{x^2+4}.$$

18. We set

$$\frac{2x^2+4x+2}{\left(x^2+1\right)^3}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{\left(x^2+1\right)^2}+\frac{Ex+F}{\left(x^2+1\right)^3}=\frac{(Ax+B)\left(x^2+1\right)^2+(Cx+D)\left(x^2+1\right)+(Ex+F)}{\left(x^2+1\right)^3}.$$

Equating numerators gives the equation $2x^2+4x+2=(Ax+B)\left(x^2+1\right)^2+(Cx+D)\left(x^2+1\right)+(Ex+F)$, or $2x^2+4x+2=Ax^5+Bx^4+(2A+C)\,x^3+(2B+D)\,x^2+(A+C+E)\,x+(B+D+F)$. Equating the coefficients of like powers of x yields the equations $A=0$, $B=0$, $2A+C=0$, $2B+D=2$, $A+C+E=4$, and $B+D+F=2$. It immediately follows that $C=0$ and $D=2$, $E=4$, and $F=0$. Thus,

$$\frac{2x^2+4x+2}{\left(x^2+1\right)^3}=\frac{2}{\left(x^2+1\right)^2}+\frac{4x}{\left(x^2+1\right)^3}.$$

19. We set

$$\frac{3x^3-5x^2+10x-19}{\left(x^2+4\right)\left(x^2+3\right)}=\frac{Ax+B}{x^2+4}+\frac{Cx+D}{x^2+3}=\frac{(Ax+B)\left(x^2+3\right)+(Cx+D)\left(x^2+4\right)}{\left(x^2+4\right)\left(x^2+3\right)}.$$

Equating numerators, we obtain $3x^3-5x^2+10x-19=(Ax+B)\left(x^2+3\right)+(Cx+D)\left(x^2+4\right)$, or $3x^3-5x^2+10x-19=(A+C)\,x^3+(B+D)\,x^2+(3A+4C)\,x+(3B+4D)$. Equating the coefficients of like powers of x yields the equations $A+C=3$, $B+D=-5$, $3A+4C=10$, and $3B+4D=-19$. We simultaneously solve the two equations in A and C to obtain $A=2$ and $C=1$. Similarly we find $B=-1$ and $D=-4$. Thus,

$$\frac{3x^3-5x^2+10x-19}{\left(x^2+4\right)\left(x^2+3\right)}=\frac{2x-1}{x^2+4}+\frac{x-4}{x^2+3}.$$

20. We set

$$\frac{2x^4+15x^2+30}{\left(x^2+4\right)\left(x^2+3\right)^2}=\frac{Ax+B}{x^2+4}+\frac{Cx+D}{x^2+3}+\frac{Ex+F}{\left(x^2+3\right)^2}$$

$$=\frac{(Ax+B)\left(x^2+3\right)^2+(Cx+D)\left(x^2+4\right)\left(x^2+3\right)+(Ex+F)\left(x^2+4\right)}{\left(x^2+4\right)\left(x^2+3\right)^2}.$$

Equating numerators, we obtain

$$2x^4+15x^2+30=(Ax+B)\left(x^2+3\right)^2+(Cx+D)\left(x^2+4\right)\left(x^2+3\right)+(Ex+F)\left(x^2+4\right).$$

Expanding the right side we obain

$$2x^4+15x^2+30=(A+C)\,x^5+(B+D)\,x^4+(6A+7C+E)\,x^3+(6B+7D+F)\,x^2$$

$$+(9A+12C+4E)\,x+(9B+12D+4F).$$

Equating the coefficients of like powers of x yields the equations $A+C=0$, $B+D=2$, $6A+7C+E=0$, $6B+7D+F=15$, $9A+12C+4E=0$, and $9B+12D+4F=30$. The first of these gives us $C=-A$ and then the third gives $E=A$. Substituting these values into the fourth equation results in $9A-12A+4A=0$, or $A=0$. Thus, $A=C=E=0$. Also, from $B+D=2$, or $D=2-B$, we obtain $6B+7(2-B)+F=15$ and $9B+12(2-B)+4F=30$. Solving these two equations simultaneously, we find that $B=2$ and $F=3$. Thus, $D=2-B=2-2=0$ and

$$\frac{2x^4+15x^2+30}{\left(x^2+4\right)\left(x^2+3\right)^2}=\frac{2}{x^2+4}+\frac{3}{\left(x^2+3\right)^2}.$$

In each of Exercises 21-30, use the Method of Partial Fractions to decompose the integrand. Then evaluate the given integral.

21. We have

$$\int\frac{3x^2-5x+4}{(x-1)\left(x^2+1\right)}dx=\int\left(\frac{1}{x-1}+\frac{2x-3}{x^2+1}\right)dx=\ln\left(|x-1|\right)+\ln\left(x^2+1\right)-3\arctan(x)+C.$$

22. We have

$$\int\frac{x^2+2}{\left(x^2+1\right)x^2}\,dx=\int\left(\frac{2}{x^2}-\frac{1}{x^2+1}\right)dx=-\frac{2}{x}-\arctan(x)+C.$$

23. We calculate

$$\int\frac{7x^3+9x-3x^2-6}{\left(x^2+2\right)\left(x^2+1\right)}\,dx=\int\left(\frac{5x}{x^2+2}+\frac{2x-3}{x^2+1}\right)dx=\frac{5}{2}\ln\left(x^2+2\right)+\ln\left(x^2+1\right)-3\arctan(x)+C.$$

In several exercises beginning with Exercise 24 it will be useful to have the following formula

$$\int \frac{2b^2}{\left(x^2+2ax+c^2\right)^2}dx = \frac{x+a}{x^2+2ax+c^2} + \frac{1}{b}\arctan\left(\frac{x+a}{b}\right) + C$$

for constants a, b, c with $b > 0$ and $c^2 = a^2 + b^2$. To obtain this formula, we write

$$x^2 + 2ax + c^2 = \left(x^2 + 2ax + a^2\right) + b^2 = (x+a)^2 + b^2,$$

we make the change of variable $x + a = b\tan(\theta)$, $dx = b\sec^2(\theta)\ d\theta$, and we calculate

$$\theta = \arctan\left(\frac{x+a}{b}\right), \qquad \sin(\theta) = \frac{x+a}{\sqrt{x^2+2ax+c^2}}, \qquad \cos(\theta) = \frac{b}{\sqrt{x^2+2ax+c^2}},$$

and

$$\int \frac{2b^2}{\left(x^2+2ax+c^2\right)^2}\,dx = \frac{2}{b}\int \frac{\sec^2(\theta)}{\sec^4(\theta)}\,d\theta = \frac{2}{b}\int \cos^2(\theta)\ d\theta = \frac{1}{b}\left(\theta + \sin(\theta)\cos(\theta)\right) + C$$

and therefore

$$\int \frac{2b^2}{\left(x^2+2ax+c^2\right)^2}\,dx = \frac{1}{b}\arctan\left(\frac{x+a}{b}\right) + \frac{x+a}{x^2+2ax+c^2} + C.$$

24. We have

$$\int \frac{x^2+2x}{\left(x^2+1\right)^2}\,dx = \int \left(\frac{2x-1}{\left(x^2+1\right)^2} + \frac{1}{x^2+1}\right)dx = \int \frac{2x}{\left(x^2+1\right)^2}\,dx - \int \frac{1}{\left(x^2+1\right)^2}\,dx + \int \frac{1}{x^2+1}\,dx.$$

Therefore, by using the formula that precedes this exercise with $a = 0$, $b = 1$, and $c = 1$, we have

$$\int \frac{x^2+2x}{\left(x^2+1\right)^2}\,dx = -\frac{1}{x^2+1} - \frac{1}{2}\left(\frac{x}{x^2+1} + \arctan(x)\right) + \arctan(x) = -\frac{1}{x^2+1} - \frac{1}{2}\frac{x}{x^2+1} + \frac{1}{2}\arctan(x) + C..$$

25. We calculate

$$\int \frac{x^3-x}{(x^2+1)^2}\,dx = \int \left(\frac{x}{x^2+1} - \frac{2x}{\left(x^2+1\right)^2}\right)dx = \int \left(\frac{1}{2}\frac{1}{x^2+1} - \frac{1}{\left(x^2+1\right)^2}\right)2x\,dx = \frac{1}{2}\ln\left(x^2+1\right) + \frac{1}{x^2+1} + C.$$

(The substitution $u = x^2 + 1$, $du = 2x\,dx$ is used to calculate the final integral. This substitution can also be applied to the original integral without resorting to partial fractions.)

26. We calculate

$$\int \frac{2x^2}{(x+1)^2(x^2+1)}\,dx = \int \left(\frac{1}{\left(x+1\right)^2} - \frac{1}{x+1} + \frac{x}{x^2+1}\right)dx = -\frac{1}{x+1} - \ln(|x+1|) + \frac{1}{2}\ln\left(x^2+1\right) + C.$$

27. We calculate

$$\int \frac{x^3+12x^2-9x+48}{(x-3)\left(x^2+4\right)}\,dx = \int \left(1 + \frac{15x^2-13x+60}{(x-3)\left(x^2+4\right)}\right)dx = \int \left(1 + \frac{12}{x-3} + \frac{3x-4}{x^2+4}\right)dx.$$

Since

$$\int \frac{3x-4}{x^2+4}\,dx = \frac{3}{2}\int \frac{2x}{x^2+4}\,dx - 4\int \frac{1}{x^2+4}\,dx = \frac{3}{2}\ln\left(x^2+4\right) - 2\arctan\left(\frac{x}{2}\right),$$

we have

$$\int \frac{x^3+12x^2-9x+48}{(x-3)\left(x^2+4\right)}\,dx = x + 12\ln(|x-3|) - 2\arctan\left(\frac{x}{2}\right) + C.$$

28. Using the formula that precedes Exercise 24 with $a = 0$, $b = 1$, and $c = 1$, we calculate

$$\int \frac{2x^2+4x+2}{(x^2+1)^3}\,dx = \int \left(\frac{2}{(x^2+1)^2} + \frac{4x}{(x^2+1)^3}\right) dx = \frac{x}{x^2+1} + \arctan(x) - \frac{1}{(x^2+1)^2} + C.$$

29. We calculate

$$\int \frac{3x^3-5x^2+10x-19}{(x^2+4)(x^2+3)}\,dx = \int \left(\frac{2x-1}{x^2+4} + \frac{x-4}{x^2+3}\right) dx$$

$$= \int \frac{2x}{x^2+4}\,dx - \int \frac{1}{x^2+4}\,dx + \frac{1}{2}\int \frac{2x}{x^2+3}\,dx - 4\int \frac{1}{x^2+3}dx.$$

Thus,

$$\int \frac{3x^3-5x^2+10x-19}{(x^2+4)(x^2+3)}\,dx = \ln\left(x^2+4\right) - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + \frac{1}{2}\ln\left(x^2+3\right) - \frac{4}{\sqrt{3}}\arctan\left(\frac{x}{\sqrt{3}}\right) + C.$$

30. Using the formula that precedes Exercise 24 with $a = 0$, $b = \sqrt{3}$, and $c = \sqrt{3}$, we calculate

$$\int \frac{2x^4+15x^2+30}{(x^2+4)(x^2+3)^2}\,dx = \int \left(\frac{2}{x^2+4} + \frac{3}{(x^2+3)^2}\right) dx = \arctan\left(\frac{1}{2}x\right) + \frac{1}{2}\frac{x}{x^2+3} + \frac{\sqrt{3}}{6}\arctan\left(\frac{x}{\sqrt{3}}\right) + C.$$

Further Theory and Practice

31. What happens if you attempt a partial fraction decomposition of

$$\frac{1}{(x^2+3)^4}$$

into

$$\frac{A_1x+B_1}{x^2+3} + \frac{A_2x+B_2}{(x^2+3)^2} + \frac{A_3x+B_3}{(x^2+3)^3} + \frac{A_4x+B_4}{(x^2+3)^4}\ ?$$

Explain.

Use the Method of Partial Fractions to calculate each of the integrals in Exercises 32-35.

32. By the Method of Partial Fractions we have

$$\frac{2x^2+4x+9}{x^3-1} = \frac{5}{x-1} - \frac{3x+4}{x^2+x+1}.$$

We write

$$x^2+x+1 = \left(x^2+x+\frac{1}{4}\right) + \frac{3}{4} = \left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2.$$

We make the change of variable

$$x + \frac{1}{2} = \frac{\sqrt{3}}{2}\tan(\theta) \qquad , \qquad dx = \frac{\sqrt{3}}{2}\sec^2(\theta)\ d\theta.$$

It follows that

$$x^2+x+1 = \left(\frac{\sqrt{3}}{2}\tan(\theta)\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \left(\frac{\sqrt{3}}{2}\right)^2\left(\tan^2(\theta)+1\right) = \left(\frac{\sqrt{3}}{2}\right)^2\sec^2(\theta)$$

and

$$\int \frac{3x+4}{x^2+x+1}\,dx = \frac{2}{\sqrt{3}}\int \frac{3\left(\frac{\sqrt{3}}{2}\tan(\theta)-\frac{1}{2}\right)+4}{\sec^2(\theta)}\sec^2(\theta)\,d\theta = 3\int \tan(\theta)\,d\theta + \frac{5}{\sqrt{3}}\int d\theta$$
$$= -3\ln\left(|\sec(\theta)|\right) + \frac{5}{\sqrt{3}}\theta + C.$$

Thus,

$$\int \frac{3x+4}{x^2+x+1}\,dx = -3\ln\left(\left|\frac{\sqrt{3}/2}{\sqrt{x^2+x+1}}\right|\right) + \frac{5}{\sqrt{3}}\theta + C$$
$$= \frac{3}{2}\ln\left(x^2+x+1\right) + \frac{5}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

and

$$\int \frac{2x^2+4x+9}{x^3-1}\,dx = 5\ln\left(|x-1|\right) - \frac{3}{2}\ln\left(x^2+x+1\right) - \frac{5}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + C.$$

33. By the Method of Partial Fractions we have

$$\frac{5x^2-2x+2}{x^3+1} = \frac{3}{x+1} + \frac{2x-1}{x^2-x+1}.$$

Thus, after making the change of variable $u = x^2-x+1$, $du = (2x-1)\,dx$ to handle the integration of the second summand, we have

$$\int \frac{5x^2-2x+2}{x^3+1}\,dx = 3\ln\left(|x+1|\right) + \ln\left(x^2-x+1\right) + C.$$

34. By the Method of Partial Fractions we have

$$\frac{8x}{x^5-x^4-x+1} = -\frac{1}{x+1} - \frac{1}{x-1} + \frac{2}{(x-1)^2} + 2\frac{x-1}{x^2+1}.$$

Thus,

$$\int \frac{8x}{x^5-x^4-x+1}\,dx = -\ln\left(|x+1|\right) - \ln\left(x-1\right) - \frac{2}{x-1} + \ln\left(x^2+1\right) - 2\arctan(x) + C.$$

35. See Exercise 31. Make the change of variable $x = \tan(\theta)$, $dx = \sec^2(\theta)\,d\theta$ to obtain $48\int \frac{\sec^2(\theta)}{\sec^8(\theta)}\,d\theta$. Write this as $48\int \cos^6(\theta)\,dx$ and use a reduction formula, as in Section 7.3. The result is

$$\int \frac{48}{(1+x^2)^4}\,dx = 8\frac{x}{(x^2+1)^3} + 10\frac{x}{(x^2+1)^2} + 15\frac{x}{x^2+1} + 15\arctan(x) + C.$$

36. At normal temperatures conduction and convection are the primary means of heat transfer. *Newton's Law of Heat Change* is reasonably accurate under these circumstances. At very high temperatures radiation is the predominant method of heat transfer. *Stefan's Law of Radiation* states that the temperature T (measured in degrees Kelvin) of an object placed in an environment at temperature T_∞ (degrees Kelvin) satisfies

$$\frac{dT}{dt} = k\left(T_\infty^4 - T^4\right)$$

for some positive constant k. By separating variables, solve this differential equation. (You will not obtain T explicitly as a function of t. Instead, you will obtain an equation that relates the two variables.)

Solution We have the partial fractions decomposition

$$\frac{4T_\infty^3}{T_\infty^4 - T^4} = -\frac{1}{T - T_\infty} + \frac{1}{T + T_\infty} + \frac{2T_\infty}{T^2 + T_\infty^2}.$$

Therefore,

$$-\ln\left(|T - T_\infty|\right) + \ln\left(|T + T_\infty|\right) + 2\arctan\left(\frac{T}{T_\infty}\right) = \int \frac{4T_\infty^3}{T_\infty^4 - T^4}\,dt = 4T_\infty^3 \int k\,dt = 4T_\infty^3 kt + C.$$

Writing $\tau = T/T_\infty$ we obtain

$$\ln\left(\left|\frac{\tau+1}{\tau-1}\right|\right) + 2\arctan\left(\tau\right) = 4T_\infty^3 kt + C.$$

CALCULATOR/COMPUTER EXERCISES

37. We solve $\left(5x^3 + 3x + 2\right) / \left(x^3 + x^2 + x + 1\right) = x^4 + 2$ and find $b = 0.758524427$. Also.

$$\frac{5x^3 + 3x + 2}{x^3 + x^2 + x + 1} - \left(x^4 + 2\right) = 3 - x^4 - \frac{3}{x+1} - \frac{2x}{x^2+1}.$$

The required area is

$$\int_0^b \left(\frac{5x^3 + 3x + 2}{x^3 + x^2 + x + 1} - \left(x^4 + 2\right)\right) dx = \int_0^b \left(3 - x^4 - \frac{3}{x+1} - \frac{2x}{x^2+1}\right) dx = 0.0774445809.$$

38. The root of $x^2 = \left(3x^3 + 4x\right) / \left(x^4 + 3x^2 + 2\right)$ for $0 \le x \le 1.1$ is $b = 1.087003363$. The required area is

$$\int_0^b \left(\frac{3x^3 + 4x}{x^4 + 3x^2 + 2} - x^2\right) dx = \int_0^b \left(\frac{x}{x^2+1} + \frac{2x}{x^2+2} - x^2\right) dx = 0.4261276278.$$

In each of Exercises 39 and 40 find all roots of the denominator of the integrand. Find a partial fractions decomposition and evaluate the given integral.

39. We factor $x^5 - 7x^4 + 4x^3 - 23x^2 + 20 = \left(x^2 + 4\right)\left(x^3 - 7x^2 + 5\right)$ and find three real roots of the cubic: $\alpha = -0.80060948993$, $\beta = 0.90578723479$, and $\gamma = 6.8948222551$. It follows that

$$\frac{2x + 4}{x^5 - 7x^4 + 4x^3 - 23x^2 + 20} = \frac{Ax + B}{x^2 + 4} + \frac{C}{x - \alpha} + \frac{D}{x - \beta} + \frac{E}{x - \gamma}.$$

We solve simultaneously for these unknowns, finding $A = 0.07111882046$, $B = 0.08673026889$, $C = 0.03936119938$, $E = -0.1179693854$, $F = 0.0074893656$. With these values, we calculate $\int_1^2 \frac{2x+4}{x^5 - 7x^4 + 4x^3 - 23x^2 + 20}\,dx \approx -0.242628758$.

40. We factor $x^5 - 2x^3 - x^2 - 3x + 1 = \left(x^2 + 0.455692986x + 1.152829370\right)(x - \alpha)(x - \beta)(x - \gamma)$ where $\alpha = -1.649139742$, $\beta = 0.2897957666$, and $\gamma = 1.815036961$. It follows that

$$\frac{3x + 1}{x^5 - 2x^3 - x^2 - 3x + 1} = \frac{Ax + B}{x^2 + 0.455692986x + 1.152829370} + \frac{C}{x - \alpha} + \frac{D}{x - \beta} + \frac{E}{x - \gamma}.$$

We solve simultaneously for these unknowns, finding $A = 0.4188089540$, $B = -0.4056398027$, $C = -0.1883031114$, $E = -0.4617804633$, $F = 0.2312746207$ With these values, we calculate $\int_2^3 \frac{3x+1}{x^5 - 2x^3 - x^2 - 3x + 1}\,dx \approx 0.245596$.

41. Computer algebra systems can quickly decompose many a fearsome rational function into its partial fractions expansion. Figure 1 shows an example of this using *Maple*. Find the partial fractions decomposition of the following rational functions:

i)

$$\frac{2x^5+13x^4-28x^3+30x^2-37x-8}{x^6-x^5+2x^4-x^3+2x^2-x+1} = \frac{-3x+14}{x^2+x+1} + \frac{5x-7}{x^2-x+1} + \frac{4x-15}{\left(x^2-x+1\right)^2}$$

ii)

$$\frac{17x^5+39x^4+140x^3+140x^2+199x+33}{x^6+3x^5+12x^4+19x^3+36x^2+27x+27} = \frac{-9x+12}{\left(x^2+x+3\right)^3} + \frac{11x-8}{\left(x^2+x+3\right)^2} + \frac{17x+5}{x^2+x+3}$$

iii)

$$\frac{2x^9+29x^7+x^6+165x^5+9x^4+431x^3+27x^2+431x+27}{x^{14}+25x^{12}+267x^{10}+1579x^8+5584x^6+11808x^4+13824x^2+6912} = \frac{2x}{\left(x^2+3\right)^3} - \frac{3x-1}{\left(x^2+4\right)^4}$$