Math 132
Fall 2005 Final Exam

1. Calculate $\int_0^{\sqrt{\pi}} x \sin(x^2)\, dx$.

a) 1 b) 2 c) 3 d) 4 e) 5

f) $\frac{1}{2}$ g) $\frac{3}{2}$ h) $\frac{5}{2}$ i) $\frac{\sqrt{2}}{2}$ j) $2\sqrt{2}$

Solution: a

```
> with(student):
> J := Int(x*sin(x^2),x = 0 .. sqrt(Pi));
```

$$J := \int_0^{\sqrt{\pi}} x \sin(x^2)\, dx$$

```
> K := changevar(x^2 = u, J, u);
```

$$K := \int_0^{\pi} \frac{1}{2} \sin(u)\, du$$

```
> value(K);
```

$$1$$

2. Let $F(x) = \int_x^{10} \frac{8+t^5}{1+t^2}\, dt$. Calculate the derivative D(F)(2) of F at 2.

a) 4 b) 5 c) 6 d) 7 e) 8

f) −4 g) −5 h) −6 i) −7 j) −8

Solution: j

```
> F := (x) -> - Int((8+t^5)/(1+t^2), t = 10 .. x);
```

$$F := x \to -\int_{10}^{x} \frac{8+t^5}{1+t^2}\, dt$$

```
> D(F)(2);
```

$$-8$$

```
> subs( t = 2, integrand(F(x)) );
```

$$8$$

3. Calculate $\displaystyle\int_{0}^{1} \frac{5x+12}{(x+2)(x+3)}\, dx$.

a) $\ln\left(\frac{4}{3}\right)$ b) $\ln\left(\frac{8}{3}\right)$ c) $\ln\left(\frac{16}{3}\right)$ d) $\ln\left(\frac{25}{3}\right)$ e) $\ln\left(\frac{28}{3}\right)$

f) $\ln\left(\frac{25}{9}\right)$ g) $\ln\left(\frac{32}{9}\right)$ h) $\ln\left(\frac{35}{6}\right)$ i) $\ln\left(\frac{35}{9}\right)$ j) $\ln\left(\frac{36}{5}\right)$

Solution: c

```
> J := Int((5*x+12)/(x+2)/(x+3),x = 0 .. 1);
```

$$J := \int_0^1 \frac{5\,x+12}{(x+2)\,(x+3)}\,dx$$

```
> A := convert(integrand(J), parfrac, x);
```

$$A := \frac{2}{x+2} + \frac{3}{x+3}$$

```
> B := int(A, x = 0 .. 1);
```

$$B := -\ln(3) + 4\ln(2)$$

```
> combine( B , ln);
```

$$-\ln\left(\frac{3}{16}\right)$$

4. Calculate $\displaystyle\int_0^1 \frac{1+3\,x^2+2\,x}{(1+x)\,(1+x^2)}\,dx$.

a) $\frac{\pi}{4}+\ln(2)$ **b)** $\frac{\pi}{4}+2\ln(2)$ **c)** $\frac{\pi}{4}+\ln(3)$ **d)** $\ln(2)$ **e)** $2\ln(2)$

f) $4\ln(2)$ **g)** $8\ln(2)$ **h)** $\ln(3)$ **i)** $2\ln(3)$ **j)** $\ln(6)$

Solution: e

```
> J := Int((1+3*x^2+2*x)/(1+x)/(1+x^2),x = 0 .. 1);
```

$$J := \int_0^1 \frac{1+3x^2+2x}{(1+x)(1+x^2)}\,dx$$

```
> A := convert( integrand(J), parfrac, x);
```

$$A := \frac{1}{1+x} + \frac{2x}{1+x^2}$$

```
> int(A, x = 0 .. 1);
```

$$2\ln(2)$$

5. Calculate $\int_0^3 \frac{1}{\sqrt{16+x^2}}\,dx$.

a) π b) $\frac{\pi}{2}$ c) $\frac{\pi}{3}$ d) $\frac{\pi}{4}$ e) $\frac{\pi}{6}$

f) $\ln(2)$ g) $\ln(3)$ h) $2\ln(2)$ i) $\ln(5)$ j) $2\ln(3)$

Solution: f

```
> J := Int(1/sqrt(16+x^2),x=0..3);
```

$$J := \int_0^3 \frac{1}{\sqrt{16+x^2}}\,dx$$

```
> K := student[changevar](x=4*tan(t), J, t);
```

$$K := \int_0^{\arctan(3/4)} \frac{4+4\tan(t)^2}{\sqrt{16+16\tan(t)^2}}\,dt$$

```
> simplify(student[integrand](K)) assuming t>0;
```

$$\sqrt{1+\tan(t)^2}$$

```
> A := int(sec(t), t);
```

$$A := \ln(\sec(t)+\tan(t))$$

```
> B := subs(t = arctan(3/4), A) - subs(t = 0, A);
```

$$B := \ln\left(\sec\left(\arctan\left(\frac{3}{4}\right)\right)+\tan\left(\arctan\left(\frac{3}{4}\right)\right)\right) - \ln(\sec(0)+\tan(0))$$

```
> simplify(B);
```

$$\ln(2)$$

6. What is the logarithmic derivative of $y(t) = 2^{(k\,t)}$?

a) k b) $\frac{1}{k}$ c) $k\,t$ d) $\frac{t}{k}$ e) $\frac{k}{\ln(2)}$

f) $k\ln(2)$ g) $\ln(k)$ h) $2\ln(k)$ i) $\ln(2)\,2^{(k\,t)}$ j) $\frac{2^{(k\,t)}}{\ln(2)}$

Solution: f

```
> diff(ln(2^(k*t)),t);
```

$$k \ln(2)$$

7. Calculate $\displaystyle\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-x^2}}\,dx$.

a) π b) $\frac{\pi}{2}$ c) $\frac{\pi}{3}$ d) $\frac{\pi}{4}$ e) $\frac{\pi}{6}$

f) $\frac{5\pi}{6}$ g) $\frac{5\pi}{12}$ h) $\frac{2\pi}{3}$ i) $\frac{3\pi}{4}$ j) $\frac{\pi}{12}$

Solution: e

```
> A := int(1/sqrt(1-x^2), x);
```

$$A := \arcsin(x)$$

```
> B := subs(x = sqrt(3)/2, A) - subs(x = 1/2, A);
```

$$B := \arcsin\left(\frac{\sqrt{3}}{2}\right) - \arcsin\left(\frac{1}{2}\right)$$

```
> simplify( B );
```

$$\frac{\pi}{6}$$

```
>
```

8. Let f$(x) = x^{(2\,x)}$. What is D(f)(1/2) ? (The derivative of f(x) at x = 1/2) ?

a) $2\ln(2)+1$ **b)** $3\ln(3)+2$ **c)** $\ln(2)+2$ **d)** $\ln(3)+3$ **e)** $\ln(2)+3$
f) $-\ln(3)+2$ **g)** $-\ln(2)+1$ **h)** $-\ln(3)+1$ **i)** $-3\ln(2)+1$ **j)** $-3\ln(3)+1$

Solution: g

```
> f := x -> x^(2*x);
```

$$f := x \to x^{(2x)}$$

```
> eqn := ln('f(x)') = ln(f(x));
```

$$eqn := \ln(f(x)) = \ln(x^{(2x)})$$

```
> eqn2 := Diff(f(x),x)/'f(x)' = diff(rhs( eqn ), x);
```

$$eqn2 := \frac{\frac{d}{dx} x^{(2x)}}{f(x)} = 2\ln(x) + 2$$

```
> eqn3 := Diff(f(x),x) = solve(eqn2, Diff(f(x),x));
```

$$eqn3 := \frac{d}{dx} x^{(2x)} = 2\ln(x)\, x^{(2x)} + 2\, x^{(2x)}$$

```
> Answer  := subs( x = 1/2, rhs(eqn3));
```

$$Answer := \ln\left(\frac{1}{2}\right) + 1$$

9. Consider the following three statements about a series $\sum_{n=1}^{\infty} a_n$ with positive terms:

I: The series converges because $\lim_{n \to \infty} a_n = 0$.

II: The series converges because $\lim_{n \to \infty} \frac{a_n}{b_n} = 1$ and $\sum_{n=1}^{\infty} b_n$ converges.

III: The series converges because $\lim_{n \to \infty} a_n^{\left(\frac{1}{n}\right)} = 1$.

For each statement, determine whether the reasoning is correct ($) or incorrect (@).

a) I: $, II: $, III: $
b) I: $, II: $, III:@
c) I: $, II: @, III: $
d) I: $, II: @, III: @
e) I:@, II: $, III: $
f) I: @, II: $, III: @

g) I: @, II: @, III: $
h) I: @, II: @, III: @
i) Wrong answer
j) Bonus wrong answer

Solution: f

10. Consider the following three statements about a series $\sum_{n=1}^{\infty} a_n$ with positive terms:

I: The series converges because $a_n < \frac{1}{n}$.

II: The series converges because $a_n < \left(\frac{n}{1+2n}\right)^n$.

III: The series converges because $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} < 1$.

For each statement, determine whether the reasoning is correct ($) or incorrect (@).

a) I: $, II: $, III: $
b) I: $, II: $, III:@
c) I: $, II: @, III: $
d) I: $, II: @, III: @
e) I:@, II: $, III: $
f) I: @, II: $, III: @
g) I: @, II: @, III: $
h) I: @, II: @, III: @
i) Wrong answer
j) Bonus wrong answer

Solution: e

11. Consider the three series

$$\text{I: } \sum_{n=2}^{\infty} \frac{n}{\ln(n)^3}, \quad \text{II: } \sum_{n=0}^{\infty} \frac{7^n}{n!}, \quad \text{and} \quad \text{III: } \sum_{n=0}^{\infty} \frac{n^2}{2^n}$$

and the statements

(\$) The series converges
(@) The series diverges

For each series, decide which of statements (\$), (&), (@) is correct.

a) I: \$, II: \$, III: \$
b) I: \$, II: \$, III:@
c) I: \$, II: @, III: \$
d) I: \$, II: @, III: @
e) I:@, II: \$, III: \$
f) I: @, II: \$, III: @
g) I: @, II: @, III: \$
h) I: @, II: @, III: @
i) Wrong answer
j) Bonus wrong answer

Solution: e

12. Consider the two series

$$\text{I: } \sum_{n=3}^{\infty} \frac{(-1)^n \ln(n)}{n^2} \quad \text{and} \quad \text{II: } \sum_{n=0}^{\infty} \frac{(-1)^n n}{(1+n)^{\left(\frac{3}{2}\right)}}$$

and the statements
(\$) The series converges absolutely
(&) The series converges conditionally
(@) The series diverges

For each series, decide which of statements (\$), (&), (@) is correct.

a) I: \$, II: \$ b) I: \$, II: &   c) I: \$, II: @   d) I: &, II: \$ e) I: &, II: &
f) I: &, II: @   g) I: @ , II: \$ h) I: @ , II: &   i) I: @ , II: @   j) Wrong answer

Solution: b

13. Consider the two series

$$\text{I: } \sum_{n=0}^{\infty} \frac{n!}{(2n)!} \quad \text{and} \quad \text{II: } \sum_{n=0}^{\infty} \frac{1}{n}$$

and the statements

(\$) The test establishes convergence

(&) The test establishes divergence

(@) The test is not conclusive.

Apply the Ratio Test to series I and the Root Test to series II. For each, decide which of statements (\$), (&), (@) is correct.

a) I: \$, II: \$ b) I: \$, II: & c) I: \$, II: @ d) I: &, II: \$ e) I: &, II: &

f) I: &, II: @ g) I: @ , II: \$ h) I: @ , II: & i) I: @ , II: @ j) Wrong answer

Solution: c

14. Let $f(x) = \sum_{n=0}^{\infty} (n-1)\left(\frac{x}{2}\right)^n$. What is $f^{(4)}(0)$?

a) 1 b) 3/2 c) 2 d) 5/2 e) 3
f) 7/2 g) 4 h) 9/2 i) 5 j) 11/2

Solution: h

```
> a := n -> (n-1)/2^n;
```

$$a := n \rightarrow \frac{n-1}{2^n}$$

```
> Answer := simplify(4!*a(4));
```

$$Answer := 24\ \mathrm{a}(4)$$

```
> p := x -> sum((n-1)*(x/2)^n,n = 0 .. 10);
  #Any upper summation index greater than 3 will work as well as
  10
```

$$p := x \rightarrow \sum_{n=0}^{10} (n-1)\left(\frac{1}{2}x\right)^n$$

```
> Direct_differentiation_answer := (D@@4)(p)(0);
```

$$\frac{9}{2}$$

15. Let $\mathrm{T}(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ be the degree 3 Taylor polynomial of $\mathrm{f}(x) = x^2 + \sin(3x)$ centered about 0. What is $\mathrm{T}\left(\frac{1}{2}\right)$?

a) 11/16 b) 3/4 c) 13/16 d) 7/8 e) 15/16
f) 17/16 g) 9/8 h) 19/16 i) 5/4 j) 21/16

Solution: h

```
> S := series(x^2+sin(3*x), x = 0, 4);
```

$$S := 3\,x + x^2 - \frac{9}{2}\,x^3 + \mathrm{O}(x^4)$$

```
> p := unapply(convert(S,polynom),x);
```

$$p := x \rightarrow 3\,x + x^2 - \frac{9}{2}\,x^3$$

```
> p(1/2);
```

$$\frac{19}{16}$$

16. Calculate the interval of convergence of $\sum_{n=0}^{\infty} \frac{(x+1)^n}{(n+1)\,2^n}$ **.**

a) $(-3, 1]$ **b)** $[-3, 1]$ **c)** $(-1, 3]$ **d)** $[-1, 3]$ **e)** $[-2, 2]$
f) $[-3, 1)$ **g)** $(-3, 1)$ **h)** $[-1, 3)$ **i)** $(-1, 3)$ **j)** $(-\infty, \infty)$

Solution: f

17. What is the coefficient of $(x-1)^3$ in the Taylor series of $x^{\left(\frac{1}{3}\right)}$ with base point 1?

a) $\frac{1}{9}$ b) $\frac{4}{27}$ c) $\frac{5}{81}$ d) $\frac{10}{243}$ e) $\frac{1}{512}$

f) $-\frac{1}{9}$ g) $-\frac{4}{27}$ h) $-\frac{5}{81}$ i) $-\frac{10}{243}$ j) $-\frac{1}{512}$

Solution: c

```
> simplify( subs(x=1,diff(x^(1/3),x$3))/3! );
```

$$\frac{5}{81}$$

```
> series(x^(1/3), x=1, 5);
```

$$1+\frac{1}{3}(x-1)-\frac{1}{9}(x-1)^2+\frac{5}{81}(x-1)^3-\frac{10}{243}(x-1)^4+\mathrm{O}((x-1)^5)$$

```
>
```

18. The Maclaurin series of $1-e^{(-x^2)}$ is used to approximate $\int_0^{0.3} 1-e^{(-x^2)}\,dx$ with an error

less than $10^{(-5)}$. The calculation uses only as many terms as are deemed necessary for the required accuracy by the Alternating Series Test. What is the approximation?

a) 0.007957 b) 0.008057 c) 0.008157 d) 0.008257 e) 0.008357
f) 0.008457 g) 0.008557 h) 0.008657 i) 0.008757 j) 0.008857

Solution: i

```
> series(1-exp(-x^2),x=0,7);
```

$$x^2 - \frac{1}{2}x^4 + \frac{1}{6}x^6 + O(x^8)$$

```
> p := convert( %, polynom );
```

$$p := x^2 - \frac{x^4}{2} + \frac{x^6}{6}$$

```
> int(subs(x=t, p), t = 0 .. x);
```

$$\frac{x^3}{3} - \frac{x^5}{10} + \frac{x^7}{42}$$

```
> subs(x=.3, 1/3*x^3-1/10*x^5);
```

$$0.008757000000$$

19. What is the coefficient of x^3 in the Maclaurin series of $\frac{25\,x}{5-2\,x}$?

a) $\frac{5}{27}$ **b)** $\frac{3}{10}$ **c)** $\frac{4}{5}$ **d)** $\frac{4}{15}$ **e)** $\frac{5}{9}$

f) $-\frac{5}{27}$ **g)** $-\frac{3}{10}$ **h)** $-\frac{4}{5}$ **i)** $-\frac{4}{15}$ **j)** $-\frac{5}{9}$

Solution: c

```
> series(25*x/(5-2*x),x=0,8);
```

$$5\,x+2\,x^2+\frac{4}{5}x^3+\frac{8}{25}x^4+\frac{16}{125}x^5+\frac{32}{625}x^6+\frac{64}{3125}x^7+O(x^8)$$

20. What is the coefficient of x^4 in the Maclaurin series of $(1+x^2)^{\left(\frac{1}{2}\right)}$?

a) $-\frac{1}{32}$ **b)** $-\frac{1}{16}$ **c)** $-\frac{1}{8}$ **d)** $-\frac{1}{4}$ **e)** $-\frac{1}{2}$

f) $\frac{1}{32}$ **g)** $\frac{1}{16}$ **h)** $\frac{1}{8}$ **i)** $\frac{1}{4}$ **j)** $\frac{1}{2}$

Solution: c

```
> binomial(1/2,2);
```

$$\frac{-1}{8}$$

```
> series((1+x^2)^(1/2), x = 0, 5);
```

$$1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + O(x^6)$$