# Math 132
# Fall 2006 Final Exam

**1. Calculate** $\int_0^2 9x^2\sqrt{1+x^3}\,dx$.

| | | | | |
|---|---|---|---|---|
| a) 20 | b) 24 | c) 28 | d) 32 | e) 36 |
| f) 40 | g) 44 | h) 48 | i) 52 | j) 56 |

## Solution: i

```
> with(student):
> J := Int(9*x^2*sqrt(1+x^3),x = 0 .. 2);
```

$$J := \int_0^2 9\,x^2\sqrt{1+x^3}\,dx$$

```
> K := changevar(1+x^3 = u, J, u);
```

$$K := \int_1^9 3\sqrt{u}\,du$$

```
> value(K);
```

$$52$$

```
> antiderivative := int(integrand(K), u);
```

$$antiderivative := 2\,u^{(3/2)}$$

```
> simplify( subs( u = 9, antiderivative) - subs( u = 1,
  antiderivative) );
```

$$52$$

**2. Let $F(x) = \int_2^x \sqrt{t^3 - 2}\,dt$. Calculate the derivative D( F )( 3 ) of F at 3.**

**a) 3 b) 4 c) 5 d) 6 e) 7**

**f) $3-\sqrt{3}$ g) $4-\sqrt{5}$ h) $5-\sqrt{6}$ i) $6-\sqrt{7}$ j) $7-2\sqrt{2}$**

## Solution: c

```
> F := (x) -> Int(sqrt(t^3-2),t = 2 .. x);
```

$$F := x \rightarrow \int_{2}^{x} \sqrt{t^3 - 2}\, dt$$

```
> D(F)(3);
```

$$\sqrt{25}$$

**3. Calculate** $\displaystyle\int_{0}^{1} \frac{x+4}{(x+1)(x+2)}\, dx$ .

a) $\ln\left(\frac{4}{3}\right)$ b) $\ln\left(\frac{8}{3}\right)$ c) $\ln\left(\frac{16}{3}\right)$ d) $\ln\left(\frac{25}{3}\right)$ e) $\ln\left(\frac{28}{3}\right)$

f) $\ln\left(\frac{25}{9}\right)$ g) $\ln\left(\frac{32}{9}\right)$ h) $\ln\left(\frac{35}{6}\right)$ i) $\ln\left(\frac{35}{9}\right)$ j) $\ln\left(\frac{36}{5}\right)$

## Solution:  g

```
> J := Int((x+4)/(x+1)/(x+2),x = 0 .. 1);
```

$$J := \int_0^1 \frac{x+4}{(x+1)(x+2)}\,dx$$

```
> processedIntegrand := convert( integrand(J), parfrac, x);
```

$$processedIntegrand := \frac{3}{x+1} - \frac{2}{x+2}$$

```
> antiderivative := int(processedIntegrand, x);
```

$$antiderivative := 3\ln(x+1) - 2\ln(x+2)$$

```
> answer := simplify(subs(x = 1, antiderivative) - subs(x = 0,
  antiderivative));
```

$$answer := 5\ln(2) - 2\ln(3)$$

```
> combine(answer, ln);
```

$$\ln\left(\frac{32}{9}\right)$$

**4. Calculate** $\displaystyle\int_1^2 \frac{5x^2+6}{(2+x^2)\,x}\,dx$.

**a)** $\frac{\pi}{4} + \ln(2)$ **b)** $\frac{\pi}{4} + 2\ln(2)$ **c)** $\frac{\pi}{4} + \ln(3)$ **d)** $\ln(2)$ **e)** $2\ln(2)$

**f)** $4\ln(2)$ **g)** $8\ln(2)$ **h)** $\ln(3)$ **i)** $2\ln(3)$ **j)** $\ln(6)$

## Solution: f

```
> J := Int( (5*x^2+6)/(2+x^2)/x, x = 1 .. 2);
```

$$J := \int_1^2 \frac{5x^2+6}{(2+x^2)\,x}\,dx$$

```
> processedIntegrand := convert( integrand(J), parfrac, x);
```

$$processedIntegrand := \frac{2x}{2+x^2} + \frac{3}{x}$$

```
> antiderivative := int(processedIntegrand, x);
```

$$antiderivative := \ln(2+x^2) + 3\ln(x)$$

```
> answer := simplify(subs(x = 2, antiderivative) - subs(x = 1,
  antiderivative));
```

$$answer := 4\ln(2)$$

**5. Calculate** $\displaystyle\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx$.

a) $\pi$ b) $\frac{\pi}{2}$ c) $\frac{\pi}{3}$ d) $\frac{\pi}{4}$ e) $\frac{\pi}{6}$

f) $2\pi$ g) $\frac{3\pi}{2}$ h) $\frac{2\pi}{3}$ i) $\frac{3\pi}{4}$ j) $\frac{5\pi}{6}$

## Solution: d

```
> J := Int(x^2/sqrt(1-x^2),x = 0 .. 1);
```

$$J := \int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx$$

```
> K := student[changevar](x=sin(t), J, t);
  #Note: denominator of K is cos(t)
```

$$K := \int_0^{\frac{\pi}{2}} \frac{\sin(t)^2\cos(t)}{\sqrt{1-\sin(t)^2}}\,dt$$

```
> int(sin(t)^2, t = 0 .. Pi/2);
```

$$\frac{\pi}{4}$$

**6. A radioactive substance has a half-life equal to ln( 8 ) years. If m(t) is the mass of the substance at time t, measured in years, then what is the logarithmic derivative of m(t) ?**

**a)** $\frac{1}{2}$ **b)** $\frac{1}{3}$ **c)** $\frac{1}{\sqrt{2}}$ **d)** $\frac{1}{\sqrt{3}}$ **e)** $\frac{1}{2\sqrt{2}}$

**f)** $-\frac{1}{2}$ **g)** $-\frac{1}{3}$ **h)** $-\frac{1}{\sqrt{2}}$ **i)** $-\frac{1}{\sqrt{3}}$ **j)** $-\frac{1}{2\sqrt{2}}$

## Solution: g

```
> k := ln(2)/ln(8); #Solve for decay constant in terms of
half-life
```

$$k := \frac{\ln(2)}{\ln(8)}$$

```
> simplify( diff( ln(A*exp(-k*t) ) , t) );
```

$$\frac{-1}{3}$$

```
> #Note: -ln(2)/ln(8) = -ln(2)/ln(2^3) = -ln(2)/(3*ln(2)) = -1/3
```

**7. Calculate** $\displaystyle\int_{1}^{\sqrt{3}} \frac{1}{1+x^2}\,dx$.

a) $\pi$ b) $\frac{\pi}{2}$ c) $\frac{\pi}{3}$ d) $\frac{\pi}{4}$ e) $\frac{\pi}{6}$

f) $\frac{5\pi}{6}$ g) $\frac{5\pi}{12}$ h) $\frac{2\pi}{3}$ i) $\frac{3\pi}{4}$ j) $\frac{\pi}{12}$

## Solution: j

```
> antiderivative := int(1/(1+x^2),x);
```

$$antiderivative := \arctan(x)$$

```
> simplify(subs(x = sqrt(3), antiderivative) - subs(x = 1,
  antiderivative));
```

$$\frac{\pi}{12}$$

**8. Let $f(x) = (1+2x)^x$. What is D( f )(1) ? (The derivative of f(x) at x = 1) ?**

**a)** $2\ln(2)+1$ **b)** $3\ln(3)+2$ **c)** $\ln(2)+2$ **d)** $\ln(3)+3$ **e)** $\ln(2)+3$
**f)** $\ln(3)+2$ **g)** $\ln(2)+1$ **h)** $\ln(3)+1$ **i)** $3\ln(2)+1$ **j)** $3\ln(3)+1$

## Solution: b

```
> f := x -> (1+2*x)^x;
```

$$f := x \to (1+2x)^x$$

```
> D(f)(1);
```

$$3\ln(3)+2$$

**9. Consider the following three statements about a series $\sum_{n=1}^{\infty} a_n$ with positive terms:**

**I: The series diverges because $\lim_{n \to \infty} a_n = 1$.**

**II: The series diverges because $\lim_{n \to \infty} \frac{a_n}{b_n} = 1$ and $\sum_{n=1}^{\infty} b_n$ diverges.**

**III: The series diverges because $\lim_{n \to \infty} a_n^{\left(\frac{1}{n}\right)} = 1$.**

**For each statement, determine whether the reasoning is correct ( $ ) or incorrect ( @ ).**

**a) I: $, II: $, III: $**
**b) I: $, II: $, III:@**
**c) I: $, II: @, III: $**
**d) I: $, II: @, III: @**
**e) I:@, II: $, III: $**
**f) I: @, II: $, III: @**
**g) I: @, II: @, III: $**
**h) I: @, II: @, III: @**
**i) Wrong answer**
**j) Bonus wrong answer**

## Solution: b

10. **Consider the following three statements about a series $\sum_{n=1}^{\infty} a_n$ with positive terms:**

**I: The series converges because** $\lim_{n \to \infty} a_n = 0$.

**II: The series converges because** $a_n < b_n$ **and** $\sum_{n=1}^{\infty} b_n$ **diverges.**

**III: The series converges because** $\lim_{n \to \infty} \frac{a_n}{a_{n+1}} < 1$.

**For each statement, determine whether the reasoning is correct ( $ ) or incorrect ( @ ).**

**a) I: $, II: $, III: $**
**b) I: $, II: $, III:@**
**c) I: $, II: @, III: $**
**d) I: $, II: @, III: @**
**e) I:@, II: $, III: $**
**f) I: @, II: $, III: @**
**g) I: @, II: @, III: $**
**h) I: @, II: @, III: @**
**i) Wrong answer**
**j) Bonus wrong answer**

## Solution: h

11. Consider the three series

$$\text{I: } \sum_{n=1}^{\infty} \frac{5^n n}{n!}, \quad \text{II: } \sum_{n=3}^{\infty} \frac{1}{n \ln(n)}, \quad \text{and} \quad \text{III: } \sum_{n=0}^{\infty} \frac{n^{\left(\frac{1}{3}\right)}}{1+n^{\left(\frac{13}{9}\right)}}$$

and the statements
( $ ) The series converges
( @ ) The series diverges

For each series, decide which of statements ($), (@) is correct.

a) I: $, II: $, III: $
b) I: $, II: $, III:@
c) I: $, II: @, III: $
d) I: $, II: @, III: @
e) I:@, II: $, III: $
f) I: @, II: $, III: @
g) I: @, II: @, III: $
h) I: @, II: @, III: @
i) Wrong answer
j) Bonus wrong answer

## Solution: c

12. Consider the two series

$$\text{I: } \sum_{n=3}^{\infty} \frac{(-1)^n}{\ln(n)} \quad \text{and} \quad \text{II: } \sum_{n=0}^{\infty} \frac{(-1)^n n^3}{2^n}$$

and the statements
( $ ) The series converges absolutely
( & ) The series converges conditionally
( @ ) The series diverges

For each series, decide which of statements ($), (&), (@) is correct.

a) I: $, II: $ b) I: $, II: & c) I: $, II: @ d) I: &, II: $ e) I: &, II: &
f) I: &, II: @ g) I: @ , II: $ h) I: @ , II: & i) I: @ , II: @ j) Wrong answer

## Solution: d

13. Consider the two series

$$\text{I: } \sum_{n=0}^{\infty} \frac{10^n}{10^n + n^2} \quad \text{and} \quad \text{II: } \sum_{n=0}^{\infty} \frac{n^3}{3^n}$$

and the statements

( $ ) The test establishes convergence

( & ) The test establishes divergence

( @ ) The test is not conclusive.

Apply the Ratio Test to series I and the Root Test to series II. For each, decide which of statements ($), (&), (@) is correct.

a) I: $, II: $ b) I: $, II: & c) I: $, II: @ d) I: &, II: $ e) I: &, II: &

f) I: &, II: @ g) I: @ , II: $ h) I: @ , II: & i) I: @ , II: @ j) Wrong answer

## Solution: g

```
> A := int(Pi*(1/Pi)*y*4,y = 1 .. 2);
```

$$A := 6$$

14. Let $f(x) = \sum_{n=1}^{\infty} \frac{(n+3)\,x^n}{n^2\,(n+1)}$. What is $f'''(0)$?

a) 1 b) 2 c) 3 d) 6 e) 12

f) 1/3 g) 1/2 h) 2/3 i) 4/3 j) 1/6

## Solution: a

```
> a := n -> (n+3)/(n^2*(n+1));
```

$$a := n \to \frac{n+3}{n^2\,(n+1)}$$

```
> simplify(3!*a(3));
```

$$1$$

```
> p := x -> sum((n+3)*x^n/(n^2*(n+1)),n = 1 .. 10);
```

$$p := x \to \sum_{n=1}^{10} \frac{(n+3)\,x^n}{n^2\,(n+1)}$$

```
> (D@@3)(p)(0);
```

$$1$$

15. Let $\mathrm{T}(x) = a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + a_4\,x^4 + a_5\,x^5$ be the degree 5 Taylor polynomial of $\mathrm{f}(x) = x^2\,\mathrm{e}^{(-2\,x)}$ centered about 0. What is $\mathrm{T}(1)$ ?

a) 1/5 b) 2/5 c) 1/3 d) 2/3 e) 4/15

f) - 1/5 g) - 2/5 h) - 1/3 i) - 2/3 j) - 4/15

## Solution: h

```
> S := series(x^2*exp(-2*x), x = 0, 6);
```

$$S := x^2 - 2\,x^3 + 2\,x^4 - \frac{4}{3}\,x^5 + \mathrm{O}(x^6)$$

```
> p := unapply(convert(S,polynom),x);
```

$$p := x \rightarrow x^2 - 2\,x^3 + 2\,x^4 - \frac{4}{3}\,x^5$$

```
> p(1);
```

$$\frac{-1}{3}$$

**16. Calculate the interval of convergence of** $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n\, n\, (x+2)^n}{(n^2+1)\, 3^n}$.

**a)** ( - 5 , 1 ]  **b)** [ - 5 , 1 ]  **c)** ( - 1 , 5 ]  **d)** [ - 1 , 5 ]  **e)** [ - 2 , 2 ]

**f)** [ - 5 , 1 )  **g)** ( - 5 , 1 )  **h)** [ - 1 , 5 )  **i)** ( - 1 , 5 )  **j)** ( $-\infty$ , $\infty$ )

## Solution: a

```
> a := n -> n/(n^2+1)/(3^n);
```

$$a := n \rightarrow \frac{n}{(n^2+1)\,3^n}$$

```
> R := limit(a(n)/a(n+1), n = infinity);
```

$$R := 3$$

**17. What is the coefficient of $(x-4)^3$ in the Taylor series of $\sqrt{x}$ with base point 4?**

a) $\frac{1}{32}$ b) $\frac{3}{64}$ c) $\frac{1}{128}$ d) $\frac{3}{256}$ e) $\frac{1}{512}$

f) $-\frac{1}{32}$ g) $-\frac{3}{64}$ h) $-\frac{1}{128}$ i) $-\frac{3}{256}$ j) $-\frac{1}{512}$

## Solution: e

```
> simplify(subs(x=4,diff(sqrt(x),x$3))/3!);
```

$$\frac{1}{512}$$

```
> series(sqrt(x), x=4, 5); #alternative
```

$$2+\frac{1}{4}(x-4)-\frac{1}{64}(x-4)^2+\frac{1}{512}(x-4)^3-\frac{5}{16384}(x-4)^4+O((x-4)^5)$$

```
>
```

**18. The Maclaurin series of $\sin(x^2)$ is used to approximate $\int_0^{0.1} 42\sin(x^2)\,dx$ with an error less than $10^{(-6)}$. The calculation uses only as many terms as are deemed necessary for the required accuracy by the Alternating Series Test. What is the approximation?**

| | | | | |
|---|---|---|---|---|
| **a) 0.012** | **b) 0.013** | **c) 0.014** | **d) 0.015** | **e) 0.016** |
| **f) 0.017** | **g) 0.018** | **h) 0.019** | **i) 0.020** | **j) 0.021** |

## Solution: c

```
> series(42*sin(x^2),x=0,16);
```

$$42\,x^2 - 7\,x^6 + \frac{7}{20}\,x^{10} - \frac{1}{120}\,x^{14} + \mathrm{O}(x^{16})$$

```
> p := convert( %, polynom );
```

$$p := 42\,x^2 - 7\,x^6 + \frac{7\,x^{10}}{20} - \frac{x^{14}}{120}$$

```
> int(subs(x=t, p), t = 0 .. x);
```

$$14\,x^3 - x^7 + \frac{7\,x^{11}}{220} - \frac{x^{15}}{1800}$$

```
> subs(x=0.1, 14*x^3);
```

$$0.014$$

```
> evalf(Int(42*sin(x^2),x = 0 .. 0.1));
```

$$0.01399990000$$

```
>
```

**19. What is the coefficient of $x^3$ in the Maclaurin series of $\frac{x}{3+2\,x}$ ?**

a) $-\frac{5}{27}$ b) $-\frac{14}{81}$ c) $-\frac{4}{27}$ d) $-\frac{10}{81}$ e) $-\frac{1}{9}$

f) $\frac{5}{27}$ g) $\frac{14}{81}$ h) $\frac{4}{27}$ i) $\frac{10}{81}$ j) $\frac{1}{9}$

## Solution: h

```
> series(x/(3+2*x),x=0,8);
```

$$\frac{1}{3}x-\frac{2}{9}x^2+\frac{4}{27}x^3-\frac{8}{81}x^4+\frac{16}{243}x^5-\frac{32}{729}x^6+\frac{64}{2187}x^7+O(x^8)$$

```
> convert( %, polynom);
```

$$\frac{x}{3}-\frac{2x^2}{9}+\frac{4x^3}{27}-\frac{8x^4}{81}+\frac{16x^5}{243}-\frac{32x^6}{729}+\frac{64x^7}{2187}$$

```
> coeff(%,x^3);
```

$$\frac{4}{27}$$

**20. What is the coefficient of $x^3$ in the Maclaurin series of $\frac{1}{(1+x)^{\left(\frac{1}{3}\right)}}$ ?**

a) $-\frac{5}{27}$ b) $-\frac{14}{81}$ c) $-\frac{4}{27}$ d) $-\frac{10}{81}$ e) $-\frac{1}{9}$

f) $\frac{5}{27}$ g) $\frac{14}{81}$ h) $\frac{4}{27}$ i) $\frac{10}{81}$ j) $\frac{1}{9}$

## Solution: b

```
> binomial(-1/3,3); #From Newton's Binomial Series
```

$$\frac{-14}{81}$$

```
> series(1/(1+x)^(1/3), x = 0, 5); #direct calculation
```

$$1-\frac{1}{3}x+\frac{2}{9}x^2-\frac{14}{81}x^3+\frac{35}{243}x^4+O(x^5)$$

```
> product(-1/3-k,k=0..2)/3!; #calculation of binomial coefficient
```

$$\frac{-14}{81}$$