9 October 2009–16 October 2009
Due 26 October 2009

1. Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ and $\mathcal{C} \subset \mathbb{R}^3$ be defined by

$$F\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x^2 + yz - 36 \\ xz + y - 17 \end{bmatrix}, \qquad \mathcal{C} = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : F\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \mathbf{0}\right\}.$$

Consider the points

$$\mathbf{a} = \begin{bmatrix} 7 \\ 31 \\ -2 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 4 \\ 1 \\ 4 \end{bmatrix}$$

on $\mathcal{C}$. For each point, use the Implicit Function Theorem to determine if the space variable z can be used to differentiably parameterize an open arc of $\mathcal{C}$ containing the point. If the answer is in the affirmative, find the parameterization. If the Implicit Function Theorem does not guarantee such a parameterization, ascertain whether there is really some impediment.

Solution We calculate $\left[D(F)\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)\right] = \begin{bmatrix} 4x & z & y \\ z & 1 & x \end{bmatrix}$. For fixed x, y, z, define $L : \mathbb{R}^2 \to \mathbb{R}^2$ by $L\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} 4x & z & y \\ z & 1 & x \end{bmatrix}\begin{bmatrix} u \\ v \\ 0 \end{bmatrix} = \begin{bmatrix} 4x & z \\ z & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$. Then L is invertible if and only if $\det\left(\begin{bmatrix} 4x & z \\ z & 1 \end{bmatrix}\right) \neq 0$. That is, L is invertible if and only if $4x \neq z^2$. This condition holds at $\mathbf{a}$ but not at $\mathbf{c}$. According to the Implicit Function Theorem, the space variable z can be used to differentiably parameterize an open arc of $\mathcal{C}$ containing $\mathbf{a}$. The Implicit Function Theorem does *not* guarantee that z can be used to differentiably parameterize an open arc of $\mathcal{C}$ containing $\mathbf{c}$. Solving $xz + y - 17 = 0$ (1.1) and $2x^2 + yz - 36 = 0$ (1.2) simultaneously, we find $y = 17 - xz$ (1.3) and, after substituting this expression for y into (1.2), we obtain $2x^2 + (17 - xz)\,z - 36 = 0$, or $2x^2 + (17 - xz)\,z - 36 = 0$ (1.4). The quadratic formula gives us

$$x = \frac{1}{4}z^2 + \frac{1}{4}\sqrt{z^4 - 136z + 288} \text{ or } x = \frac{1}{4}z^2 + \frac{1}{4}\sqrt{z^4 - 136z + 288}.$$

At point $\mathbf{a}$ we have $z = -2$ and $x = 7 > \frac{1}{4}(-2)^2 = \frac{1}{4}z^2$. Therefore we use the positive sign to get the formula for x in terms of z. If we substitute this into (1.3) we obtain the formula for y. The parameterization is

$$\begin{aligned} x &= \tfrac{1}{4}z^2 + \tfrac{1}{4}\sqrt{z^4 - 136z + 288} \\ y &= 17 - \left(\tfrac{1}{4}z^2 + \tfrac{1}{4}\sqrt{z^4 - 136z + 288}\right) z \end{aligned}$$

for z in a sufficiently small interval centered at -2. Turning to $\mathbf{c}$, we examine

$$x = \frac{1}{4}z^2 + \frac{1}{4}\sqrt{z^4 - 136z + 288} = \frac{1}{4}z^2 + \frac{1}{4}\sqrt{(z-4)\left(z^3 + 4z^2 + 16z - 72\right)}.$$

If $z < 4$, then $(z-4)\left(z^3 + 4z^2 + 16z - 72\right) = (-)(+) < 0$ and the formula we found for x is not defined on an interval *centered* at 4: it is only defined to the right of 4. That in itself would rule out the existence of a parameterization by z on an *open* arc containing $\mathbf{c}$. Additionally, there is another problem at $\mathbf{c}$. :

$$\frac{d}{dz}\left(\frac{1}{4}z^2 + \frac{1}{4}\sqrt{z^4 - 136z + 288}\right) = \frac{1}{2}z + \frac{4z^3 - 136}{8\sqrt{z^4 - 136z + 288}}$$

and the denominator is 0 when $z = 4$. Thus, even if the parameterization eztended to each side of 4, the parameterization would not be differentiable at 4.

2. Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ and $\mathcal{C} \subset \mathbb{R}^3$ be defined by

$$F\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} xy + z + yz - 5 \\ x + y + z^2 - 4 \end{bmatrix}, \qquad \mathcal{C} = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : F\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \mathbf{0}\right\}.$$

Consider the point

$$\mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

on $\mathcal{C}$. For each space variable x, y, z, use the Implicit Function Theorem to determine whether the variable can be used to differentiably parameterize an open arc of $\mathcal{C}$ containing $\mathbf{c}$. For each variable, if the answer is in the affirmative , find the parameterization—provided that the parameterization does not involve the solution of a cubic equation. Or, if the Implicit Function Theorem does not guarantee such a parameterization, ascertain whether there is really some impediment.

Solution We calculate $\left[D(F)\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)\right] = \begin{bmatrix} y & x+z & 1+y \\ 1 & 1 & 2z \end{bmatrix}$. Thus, $[D(F)(\mathbf{c})] = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$. Define $L : \mathbb{R}^2 \to \mathbb{R}^2$ by $L\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$, which is not invertible. (The determinant is 0). The Implicit Function Theorem does *not* guarantee that z can be used as a parameter on an open arc containing $\mathbf{c}$. If there were such a parameterization, we would have $y = 4 - x - z^2$ (2.1) and $xy + z + yz - 5 = 0$ (2.2). Substituting (2.1) into (2.2) yields $x\left(4 - x - z^2\right) + z + \left(4 - x - z^2\right)z - 5 = 0$, or, on solving for x, $x\left(4 - x - z^2\right) + z + \left(4 - x - z^2\right)z - 5 = 0$, either $x = 2 - \frac{1}{2}z^2 - \frac{1}{2}z - \frac{1}{2}\sqrt{z^4 - 2z^3 - 7z^2 + 12z - 4}$ or$x = 2 - \frac{1}{2}z^2 - \frac{1}{2}z + \frac{1}{2}\sqrt{z^4 - 2z^3 - 7z^2 + 12z - 4}$. Neither expression is differentiable at $z = 1$ since $\sqrt{z^4 - 2z^3 - 7z^2 + 12z - 4} = \sqrt{(z-1)\left(z^3 - z^2 - 8z + 4\right)}$. There are two problems here (and just one of the problems would be fatal). The radical is defined only for $z \le 1$, so $\mathbf{c}$ would not be interior to an open arc. Furthermore, x is not a differentiable function of z at $z = -1$ since

$$\frac{d}{dz}\sqrt{z^4 - 2z^3 - 7z^2 + 12z - 4} = \frac{2z^3 - 3z^2 - 7z + 6}{\sqrt{z^4 - 2z^3 - 7z^2 + 12z - 4}}$$

and the denominator is 0 for $z = 1$. Next, define $L : \mathbb{R}^2 \to \mathbb{R}^2$ by $L\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} u \\ 0 \\ v \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$, which *is* invertible, the determinant being nonzero. The Implicit Function Theorem tells us that y can be used to differentiably parameterize an open arc of $\mathcal{C}$ containing $\mathbf{c}$. From (2.1) we get $x = 4 - y - z^2$ (2.3) and from substituting (2.3) into (2.2), we get $\left(4 - y - z^2\right)y + z + yz - 5 = 0$, or

$$z = -\frac{1}{2y}\left(-1 - y \pm \sqrt{1 - 18y + 17y^2 - 4y^3}\right).$$

If we note that $z = 1$ when $y = 2$, we see that we must use the positive sign. Thus,

$$z = -\frac{1}{2y}\left(-1 - y + \sqrt{1 - 18y + 17y^2 - 4y^3}\right) \qquad \text{and} \qquad x = 4 - y - \left(-\frac{1}{2y}\left(-1 - y + \sqrt{1 - 18y + 17y^2 - 4y^3}\right)\right)^2$$

for y in some open interval containing 2. This is the required parameterization. Finally, define $L : \mathbb{R}^2 \to \mathbb{R}^2$ by $L\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ u \\ v \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$, which *is* invertible, the determinant being nonzero. The Implicit Function Theorem tells us that x can be used to differentiably parameterize an open arc of $\mathcal{C}$ containing $\mathbf{c}$. From (2.1) we get $y = 4 - x - z^2$ and from substituting this into (2.2), we get $x\left(4 - x - z^2\right) + z + \left(4 - x - z^2\right)z - 5 = 0$. Solving for z in terms of x amounts to solving a cubic, so it need not be done.

3. Let $F : \mathbb{R}^3 \to \mathbb{R}$ and $\mathcal{S} \subset \mathbb{R}^3$ be defined by

$$F\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = 2x^2 + xy + yz, \qquad \mathcal{S} = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : F\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = 0\right\}.$$

Consider the points

$$\mathbf{a} = \begin{bmatrix} 2 \\ -8 \\ -1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{c} = \begin{bmatrix} 2 \\ 2 \\ -6 \end{bmatrix}$$

on $\mathcal{S}$. For each point, use the Implicit Function Theorem to determine which pairs of space variables, $\{x, y\}$, $\{x, z\}$, $\{y, z\}$, can be used to differentiably parameterize an open patch of $\mathcal{S}$. For each pair of variables, if the answer is in the affirmative, find the parameterization. Or, if the Implicit Function Theorem does not guarantee such a parameterization, ascertain whether there is really some impediment.

Solution We calculate $\left[D(F)\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)\right] = \begin{bmatrix} 4x+y & x+z & y \end{bmatrix}$. Thus $\mathbf{a}$, $[D(F)(\mathbf{a})] = \begin{bmatrix} 0 & 1 & -8 \end{bmatrix}$ and $[D(F)(\mathbf{c})] = \begin{bmatrix} 10 & -4 & 2 \end{bmatrix}$. Let us consider $\mathbf{c}$ first. The three maps $L : \mathbb{R} \to \mathbb{R}$ defined by $L([w]) = \begin{bmatrix} 10 & -4 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ w \end{bmatrix} = 2w$, $L : \mathbb{R} \to \mathbb{R}$ defined by by $L([v]) = \begin{bmatrix} 10 & -4 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ v \\ 0 \end{bmatrix} = -4v$, and $L([u]) = \begin{bmatrix} 10 & -4 & 2 \end{bmatrix} \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix} = 10u$ are all invertible. Thus, each pair of space variables, $\{x, y\}$, $\{x, z\}$, $\{y, z\}$, can be used to differentiably parameterize an open patch of $\mathcal{S}$ around $\mathbf{c}$. The parameterization by x and y is $z = -\frac{2x^2+xy}{y}$. The parameterization by x and z is $y = -2\frac{x^2}{x+z}$. The parameterization by y and z is $x = -\frac{1}{4}y + \frac{1}{4}\sqrt{y^2 - 8yz}$. We turn to the point $\mathbf{a}$ now. Of the three maps $L : \mathbb{R} \to \mathbb{R}$ defined by $L([w]) = \begin{bmatrix} 0 & 1 & -8 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ w \end{bmatrix} = -8w$, $L : \mathbb{R} \to \mathbb{R}$ defined by by $L([v]) = \begin{bmatrix} 0 & 1 & -8 \end{bmatrix} \begin{bmatrix} 0 \\ v \\ 0 \end{bmatrix} = v$, and $L([u]) = \begin{bmatrix} 0 & 1 & -8 \end{bmatrix} \begin{bmatrix} u \\ 0 \\ 0 \end{bmatrix} = 0$, only the first two are invertible. Thus pairs of space variables, $\{x, y\}$ and $\{x, z\}$ can both be used to differentiably parameterize an open patch of $\mathcal{S}$ around $\mathbf{c}$. The parameterization by x and y is $z = -\frac{2x^2+xy}{y}$. The parameterization by x and z is $y = -2\frac{x^2}{x+z}$. The parameterization by y and z would have to be $x = -\frac{1}{4}y \pm \frac{1}{4}\sqrt{y^2 - 8yz}$, but because the radical vanishes at $\mathbf{a}$, x is not differentiable at $\mathbf{a}$.

4. Show that

$$\mathcal{M} = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \setminus \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\} : -x^2 + y^2 + z^2 = 0\right\}$$

is a smooth 2-manifold in $\mathbb{R}^3$. Do this in three different ways: the local graph approach, the local solution set approach, and the local parameterization approach.

Solution If $\begin{matrix} x \\ y \\ z \end{matrix} \in \mathcal{M}$ with $0 < x$, then $x = \sqrt{y^2 + z^2}$ and this is a differentiable function. If $\begin{matrix} x \\ y \\ z \end{matrix} \in \mathcal{M}$ with $x < 0$, then $x = -\sqrt{y^2 + z^2}$ and this is a differentiable function. Thus, $\mathcal{M}$ is locally the graph of a differentiable function. This is one way to see that $\mathcal{M}$ is a smooth 2-manifold in $\mathbb{R}^3$. Next, let $F\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = -x^2 + y^2 + z^2$.

Then F is differentiable and $\left[D(F)\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)\right] = \begin{bmatrix} -2x & 2y & 2z \end{bmatrix}$. Thus, $D(F)\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)\left(\begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = -2xu + 2yv + 2zw$. We want to show that $D(F)\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) : \mathbb{R}^3 \to \mathbb{R}$ is onto for every $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathcal{M}$. That is, we want to show that if $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is any point of $\mathcal{M}$ and if α is any real number, then we can find $\begin{bmatrix} u \\ v \\ w \end{bmatrix}$ for which $D(F)\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)\left(\begin{bmatrix} u \\ v \\ w \end{bmatrix}\right) = \alpha$. Notice that at every point $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ of $\mathcal{M}$, at least one of x, y, z is nonzero. If $x \neq 0$, then $D(F)\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)\left(\begin{bmatrix} -\alpha/(2x) \\ 0 \\ 0 \end{bmatrix}\right) = -2x(-\alpha/(2x)) + 2y(0) + 2z(0) = \alpha$. If $y \neq 0$, then $D(F)\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)\left(\begin{bmatrix} 0 \\ \alpha/(2y) \\ 0 \end{bmatrix}\right) = -2x(0) + 2y(\alpha/(2y)) + 2z(0) = \alpha$. If $z \neq 0$, then $D(F)\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)\left(\begin{bmatrix} 0 \\ 0 \\ \alpha/(2z) \end{bmatrix}\right) = -2x(0) + 2y(0) + 2z(\alpha/(2z)) = \alpha$. Thus, $D(F)\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$ is onto for every $\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) \in \mathcal{M}$. That is the second way to show that $\mathcal{M}$ is a smooth manifold. Finally, we will produce a smooth parameterization of $\mathcal{M}$. One way to do this is to slice $\mathcal{M}$ with planes of the form $x =$constant. The slices are circles of radius x and so $\gamma\left(\begin{bmatrix} x \\ t \end{bmatrix}\right) = \begin{bmatrix} x \\ x\cos(t) \\ x\sin(t) \end{bmatrix}$ is one way to parameterize. Another approach is to slice with $z =$constant planes. These slices are hyperbolas. For this parameterization we take $\gamma\left(\begin{bmatrix} z \\ t \end{bmatrix}\right) = \begin{bmatrix} z\cosh(t) \\ z\sinh(t) \\ z \end{bmatrix}$. Whichever parameterization γ we use, we must show that $D(\gamma)(p)$ is 1-1. For example,

$$\left[D(\gamma)\left(\begin{bmatrix} x \\ t \end{bmatrix}\right)\right] = \begin{bmatrix} 1 & 0 \\ \cos(t) & -x\sin(t) \\ \sin(t) & x\cos(t) \end{bmatrix}$$

and

$$D(\gamma)\left(\begin{bmatrix} x \\ t \end{bmatrix}\right)\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ \cos(t) & -x\sin(t) \\ \sin(t) & x\cos(t) \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u \\ u\cos(t) - xv\sin(t) \\ u\sin(t) + xv\cos(t) \end{bmatrix}.$$

If this is $\mathbf{0}$, then we obtain $u = 0$ immediately. The equations $u\cos(t) - xv\sin(t) = 0$ and $u\sin(t) + xv\cos(t) = 0$ become $xv\sin(t) = 0$ and $xv\cos(t) = 0$. It follows that $0 = 0^2 + 0^2 = (xv\sin(t))^2 + (xv\cos(t))^2 = x^2v^2$. Since $x \neq 0$, we see that $v = 0$. Thus, the kernel (nullspace) of $D(\gamma)\left(\begin{bmatrix} x \\ t \end{bmatrix}\right)$ is (0) and so $D(\gamma)\left(\begin{bmatrix} x \\ t \end{bmatrix}\right)$ is 1-1. This completes the third method of showing that $\mathcal{M}$ is a smooth manifold.

5. Show that

$$\mathcal{M} = \left\{\begin{bmatrix} t\cos(\pi t) \\ t\sin(\pi t) \\ t \end{bmatrix} \in \mathbb{R}^3 : 0 < t < 1\right\}$$

is a smooth 1-manifold in $\mathbb{R}^3$. Do this in three different ways: the local graph approach, the local solution set approach, and the local parameterization approach.

Solution We start with the parameterization $\gamma(t) = \begin{bmatrix} t\cos(\pi t) \\ t\sin(\pi t) \\ t \end{bmatrix}$ of $\mathcal{M}$, which is given. We calculate $D(\gamma)(t)\left(\begin{bmatrix} u \end{bmatrix}\right) = \begin{bmatrix} \cos(\pi t) - \pi t\sin(\pi t) \\ \sin(\pi t) + \pi t\cos(\pi t) \\ 1 \end{bmatrix}\begin{bmatrix} u \end{bmatrix} = \begin{bmatrix} u(\cos(\pi t) - \pi t\sin(\pi t)) \\ u(\sin(\pi t) + \pi t\cos(\pi t)) \\ u \end{bmatrix}$. Clearly, if $D(\gamma)(t)\left(\begin{bmatrix} u \end{bmatrix}\right) = \mathbf{0}$ then $u = 0$. This tells us that $D(\gamma)(t)$ is 1-1 for every t, which is the required condition for a parameterization. Clearly $\mathcal{M}$ is the graph of $f : (0,1) \to \mathbb{R}^2$ defined by $f(t) = \begin{bmatrix} t\cos(\pi t) \\ t\sin(\pi t) \end{bmatrix}$. Since f is differentiable, this representation of $\mathcal{M}$ gives us a second way of seeing that it is a smooth manifold. Finally, let us find two equations among the space variables $x = t\cos(\pi t)$, $y = t\sin(\pi t)$, and $z = t$. We see that $x^2 + y^2 = t^2 = z^2$. Also, since $t\sin(\pi t) \neq 0$ for $0 < t < 1$, we see that $y \neq 0$ on $\mathcal{M}$. We can therefore divide by y to obtain have $x/y = \cot(\pi t) = \cot(\pi z)$. If $F\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \left(\begin{bmatrix} x^2 + y^2 - z^2 \\ x/y - \cot(\pi z) \end{bmatrix}\right)$, then for each $p \in \mathcal{M}$ there is a $r > 0$ such that $\mathcal{M} \cap B_r(p) = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : F\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \mathbf{0}\right\}$. We calculate

$$\left[D(F)\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)\right] = \begin{bmatrix} 2x & 2y & -2z \\ 1/y & -x/y^2 & \pi\csc^2(\pi z) \end{bmatrix}.$$

To show that the derivative of F is onto, we must show that, for any pair of real numbers α and β, we can find u, v, and w such that

$$\begin{bmatrix} 2x & 2y & -2z \\ 1/y & -x/y^2 & \pi\csc^2(\pi z) \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

Observe that, since $y > 0$ on $\mathcal{M}$, we have $\det\left(\begin{bmatrix} 2x & 2y \\ 1/y & -x/y^2 \end{bmatrix}\right) = -2\frac{x^2+y^2}{y^2} < 0$ (and, in particular, is not 0). Therefore, the linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} 2x & 2y & -2z \\ 1/y & -x/y^2 & \pi\csc^2(\pi z) \end{bmatrix}\begin{bmatrix} u \\ v \\ 0 \end{bmatrix} = \begin{bmatrix} 2x & 2y \\ 1/y & -x/y^2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

is invertible, hence onto. Thus, for every pair of real numbers α and β, we can find u and v such that

$$\begin{bmatrix} 2x & 2y & -2z \\ 1/y & -x/y^2 & \pi\csc^2(\pi z) \end{bmatrix}\begin{bmatrix} u \\ v \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

This solves the required equation with $w = 0$.