Math 318 Fall 2008
(Practice) Exam 2

1. The set
$$\left\{(x,y,z)\in\mathbb{R}^3:\sqrt{\frac{x^2}{z}+\frac{2y^2}{z}}=3,(x-1)^2+(y-2)^2+(z-1)^2<\frac{1}{10000}\right\}$$
is a smooth manifold M embedded in $\mathbb{R}^3$. Represent M as a manifold in three different ways—one way has essentially been given to you. For each representation, state explicitly the function(s) that is(are) used to define M. Verify the condition(s) required of the derivative(s) of the defining function(s). (Save your derivatives for the next problem.)

Solution Let $F:\mathbb{R}^3\to\mathbb{R}$ be defined by $F(x,y,z)=x^2+2y^2-9z$. Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by $f(x,y)=\left(x^2+2y^2\right)/9$. Let $\gamma:\mathbb{R}^2\to\mathbb{R}^3$ be defined by $\gamma(s,t)=\left(s,t,\left(s^2+2t^2\right)/9\right)$. Let $a=(1,2)$ and $c=(1,2,1)$. Set $U=B_{1/100}(c)$. Then $M=\left\{(x,y,z)\in\mathbb{R}^3:F(x,y,z)=0\right\}\cap U$, $M=\Gamma(f)\cap U$, and $M=\left\{(x,y,z)\in\mathbb{R}^3:[x,y,z]=\gamma(x,y)\right\}\cap U$. Observe that $D(F)\left(\overrightarrow{\mathbf{x}}\right):\mathbb{R}^3\to\mathbb{R}$ is given by
$$D(F)\left(\overrightarrow{\mathbf{x}}\right)\left[\overrightarrow{\mathbf{u}}\right]=\begin{bmatrix}2x & 4y & -9\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}=2xu+4yv-9w \text{ where } x=\begin{bmatrix}x\\y\\z\end{bmatrix} \text{ and } \overrightarrow{\mathbf{u}}=\begin{bmatrix}u\\v\\w\end{bmatrix}.$$
Therefore, if t is an arbitrary real number, by setting $x=y=0$ and $w=-t/9$, we have $D(F)\left(\overrightarrow{\mathbf{x}}\right)\left[\overrightarrow{\mathbf{u}}\right]=t$. Thus $D(F)\left(\overrightarrow{\mathbf{x}}\right)$ is onto for $\overrightarrow{\mathbf{x}}\in M$. Using the function f, the only requirement is that f be a continuously differentiable function of x and y for x and y such that $\begin{bmatrix}x\\y\\f(x,y)\end{bmatrix}\in U$. Since $D_1(f)\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=2x/9$ and $D_2(f)\left(\begin{bmatrix}x\\y\end{bmatrix}\right)=4y/9$, there is clearly no problem. Finally, using the parameterization γ, we must show that $D(\gamma)\left(\begin{bmatrix}s\\t\end{bmatrix}\right)\mathbb{R}^2\to\mathbb{R}^3$ is 1-1. Since
$$D(\gamma)\left(\begin{bmatrix}s\\t\end{bmatrix}\right)\begin{bmatrix}u\\v\end{bmatrix}=\begin{bmatrix}1&0\\0&1\\2s/9&4t/9\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}=\begin{bmatrix}u\\v\\(2su+4tv)/9\end{bmatrix}$$
we see that $D(\gamma)\left(\begin{bmatrix}s\\t\end{bmatrix}\right)$ is 1-1 for all $\begin{bmatrix}s\\t\end{bmatrix}$.

2. Let M be the manifold of the preceding problem. Calculate the tangent space $T_{\overrightarrow{\mathbf{c}}}M$ where $\overrightarrow{\mathbf{c}}=(1,2,1)$. Demonstrate that your asserted $T_{\overrightarrow{\mathbf{c}}}M$ is in agreement with what theory states T_cM must be for each of the three representations of M.

Solution There are three ways to represent the tangent space T_cM: one for each representation of M. Using F, we have $D(F)\left(\overrightarrow{\mathbf{c}}\right)\begin{bmatrix}u\\v\\w\end{bmatrix}=\begin{bmatrix}2&8&-9\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}=2u+8v-9w$. Since $T_{\overrightarrow{\mathbf{c}}}M=\ker\left(D(F)\left(\overrightarrow{\mathbf{c}}\right)\right)$, we have
$$T_{\overrightarrow{\mathbf{c}}}M=\left\{\begin{bmatrix}u\\v\\w\end{bmatrix}\in\mathbb{R}^3:2u+8v-9w=0\right\}.$$
Using f, we have $D(f)\left(\begin{bmatrix}1\\2\end{bmatrix}\right)=[2/9,8/9]$. Since $T_{\overrightarrow{\mathbf{c}}}M=\Gamma\left(D(f)\left(\overrightarrow{\mathbf{c}}\right)\right)$, we have
$$T_{\overrightarrow{\mathbf{c}}}M=\left\{\begin{bmatrix}x\\y\\z\end{bmatrix}\in\mathbb{R}^3:z=\frac{2}{9}x+\frac{8}{9}y\right\}.$$

Since the equation $z = 2x/9 + 8y/9$ is equivalent to $2u + 8v - 9w = 0$, we see that the same tangent space results. Using γ, we have

$$D(\gamma)\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2/9 & 8/9 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u \\ v \\ (2u+8v)/9 \end{bmatrix}.$$

Thus, $T_{\overrightarrow{\mathbf{c}}}M = \Gamma\left(D(f)\left(\overrightarrow{\mathbf{c}}\right)\right)$ is given by

$$T_{\overrightarrow{\mathbf{c}}}M = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x = u, y = v, z = (2u+8v)/9 \text{ for some } \begin{bmatrix} u \\ v \end{bmatrix} \in \mathbb{R}^2\right\}.$$

If $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a point in this representation of $T_{\overrightarrow{\mathbf{c}}}M$, then

$$2x + 8y - 9z == 2(u) + 8(v) - 9\left(\frac{2u+8v}{9}\right) = 0.$$

So this representation agrees with the others.

3. Calculate the degree 2 Taylor polynomial of $f(x,y,z) = x^2 + y/z$ with base point $c = (-1,2,1)$.

Solution We calculate $f(-1,2,1) = 3$, $D_{(1,0,0)}(f)(x,y,z) = 2x$, $D_{(0,1,0)}(f)(x,y,z) = 1/z$, $D_{(0,0,1)}(f)(x,y,z) = -y/z^2$, $D_{(2,0,0)}(f)(x,y,z) = 2$, $D_{(0,2,0)}(f)(x,y,z) = 0$, $D_{(0,0,2)}(f)(x,y,z) = 2y/z^3$, $D_{(1,1,0)}(f)(x,y,z) = 0$, $D_{(1,0,1)}(f)(x,y,z) = 0$, $D_{(0,1,1)}(f)(x,y,z) = -1/z^2$, and so $D_{(1,0,0)}(f)(-1,2,1) = -2$, $D_{(0,1,0)}(f)(-1,2,1) = 1$, $D_{(0,0,1)}(f)(-1,2,1) = -2$, $D_{(2,0,0)}(f)(-1,2,1) = 2$, $D_{(0,2,0)}(f)(-1,2,1) = 0$, $D_{(0,0,2)}(f)(-1,2,1) = 4$, $D_{(1,1,0)}(f)(-1,2,1) = 0$, $D_{(1,0,1)}(f)(-1,2,1) = 0$, $D_{(0,1,1)}(f)(-1,2,1) = -1$. The required polynomial is

$$P_{f,c,2}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = 3 - 2(x+1) + (y-2) - 2(z-1) + \frac{1}{2}2(x+1)^2 + \frac{1}{2}4(z-1)^2 - (y-2)(z-1).$$

or

$$P_{f,c,2}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = 2 + 2y - 4z + x^2 + 2z^2 - yz.$$

4. Find the degree 3 Taylor polynomial of $f(x,y) = (\cos(x+y))/(1-y)$ with base point $(0,0)$.

Solution We have from known series

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{\cos(x+y)}{1-y} = \left(1 - \frac{(x+y)^2}{2} + \frac{(x+y)^4}{24} - \cdots\right)(1 + y + y^2 + y^3 + \cdots)$$

or

$$P_{f,\mathbf{0},3}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 1 + y + y^2 + y^3 - \frac{(x+y)^2}{2} \cdot y$$

or

$$P_{f,\mathbf{0},3}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 1 + y + y^2 - \frac{1}{2}x^2y - xy^2 + \frac{1}{2}y^3.$$

5. What is the signature of $Q(x,y,z) = x^2 + 2xy - 4xz$?

Solution We calculate

$$Q(x,y,z) = x^2 + 2xy - 4xz = x^2 + 2(y-2z)x = \left(x^2 + 2(y-2z)x + (y-2z)^2\right) - (y-2z)^2$$

or

$$Q(x,y,z) = (x+y-2z)^2 - (y-2z)^2.$$

If $A(x+y-2z) + B(y-2z) = 0$ for all x, y, and z, then for $y = z = 0$ and $x = 1$ we obtain $A = 0$. It follows that $B(y-2z) = 0$ for all y and z. Setting $y = 3$ and $z = 1$ we obtain $B = 0$. Thus, the linear functions

$$L_1\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = x + y - 2z \text{ and } L_2\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = y - 2z$$

are linearly independent. It follows that Q has signature $(1,1)$.

6. Find and classify the critical points of $f(x,y) = \left(x^2+y^2\right)\exp\left(x^2-y^2\right)$.

Solution We have

$$D_1(f)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 2x\exp\left(x^2-y^2\right) + 2\left(x^2+y^2\right)x\exp\left(x^2-y^2\right) = 2x\left(1+x^2+y^2\right)\exp\left(x^2-y^2\right)$$

and

$$D_2(f)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 2y\exp\left(x^2-y^2\right) - 2\left(x^2+y^2\right)y\exp\left(x^2-y^2\right) = 2y\left(1-x^2-y^2\right)\exp\left(x^2-y^2\right).$$

Setting these derivatives equal to 0, we obtain the simultaneous equations

$$2x\left(1+x^2+y^2\right) = 0 \text{ and } 2y\left(1-x^2-y^2\right).$$

From the first we see that $x = 0$. From the second we see that $y = -1, 0$, or 1. There are three critical points $P = (0,-1)$, $Q = (0,0)$, and $R = (0,1)$. We (reluctantly) calculate

$$D_{(2,0)}(f)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \left(2+10x^2+2y^2+4x^4+4x^2y^2\right)\exp\left(x^2-y^2\right),$$

$$D_{(0,2)}(f)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \left(2-10y^2-2x^2+4x^2y^2+4y^4\right)\exp\left(x^2-y^2\right),$$

and

$$D_{(1,1)}(f)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = -4\left(x^2+y^2\right)xy\exp\left(x^2-y^2\right).$$

Therefore

$$\operatorname{Discr}\left(f, \begin{bmatrix} x \\ y \end{bmatrix}\right) = \left(4+16x^2-16y^2-12x^4-8x^6-88x^2y^2-12y^4+8y^6-8x^4y^2+8x^2y^4\right)\exp\left(2\left(x^2-y^2\right)\right)$$

and in particular

$$\operatorname{Discr}\left(f, \begin{bmatrix} 0 \\ y \end{bmatrix}\right) = \left(4-16y^2-12y^4+8y^6\right)\exp\left(-2y^2\right).$$

Notice that Discr(P) =Discr(R) $= (4-16-12+8)\exp(-2) = -16\exp(-2) < 0$. Therefore P and R are saddle points. Also, Discr$\left(f, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = 4 > 0$, so a local extremum occurs at Q. Because $D_{(2,0)}(f)(Q) = 2 > 0$, we deduce that the extremum is a local minimum.

7. Let $S = \left\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\right\}$. Let $f : S \to \mathbb{R}$ be defined by $f(x,y,z) = x + 2y + 2z$. Investigate the extrema of f.

Solution Letting $g(x,y,z)=x^2+y^2+z^2-1$, we set $\nabla(f)\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)=\lambda\nabla(g)\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$, or $1=2\lambda x$, $2=2\lambda y$, $2=2\lambda z$. Notice that $\lambda\neq 0$. Thus $x=1/(2\lambda)$, $y=1/\lambda$, and $z=1/\lambda$. It follows that $(1/(2\lambda))^2+(1/\lambda)^2+(1/\lambda)^2=1$, which gives us $\lambda=-3/2$ or $\lambda=3/2$. The critical points are $(-1/3,-2/3,-2/3)$ and $(1/3,2/3,2/3)$. The first is a minimum and the second is a maximum.

8. Let C be the circle in $\mathbb{R}^3$ that is obtained by intersecting the unit sphere S of the preceding problem with the plane $\{(x,y,z)\in\mathbb{R}^3 : 2x-y+z=0\}$. Investigate the extrema that result when the function f of the preceding problem is restricted to C.

Solution Letting $g(x,y,z)=x^2+y^2+z^2-1$ and $h(x,y,z)=2x-y+z$, we set

$$\nabla(f)\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)=\lambda\nabla(g)\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)+\mu\nabla(h)\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right),$$

or $1=2\lambda x+2\mu$, $2=2\lambda y-\mu$, $2=2\lambda z+\mu$. Notice that $\lambda\neq 0$. Thus $x=(1-2\mu)/(2\lambda)$, $y=(2-\mu)/(2\lambda)$, and $z=(2-\mu)/(2\lambda)$. It follows that

$$\left(\frac{1-2\mu}{2\lambda}\right)^2+\left(\frac{2-\mu}{2\lambda}\right)^2+\left(\frac{2-\mu}{2\lambda}\right)^2=1,$$

or

$$\frac{9-12\mu+6\mu^2}{4\lambda^2}=1, \qquad (*)$$

and

$$2\left(\frac{1-2\mu}{2\lambda}\right)-\left(\frac{2-\mu}{2\lambda}\right)+\left(\frac{2-\mu}{2\lambda}\right)=0$$

or

$$\frac{1-2\mu}{2\lambda}=0.$$

This last equation tells us that $\mu=1/2$. Then $(*)$ becomes $9-6+6/4=4\lambda^2$, or $\lambda=\pm\frac{3}{4}\sqrt{2}$. The minus sign gives the critical point $\left(0,-\frac{1}{2}\sqrt{2},-\frac{1}{2}\sqrt{2}\right)$, which is a minimum, and the plus sign gives the critical point $\left(0,\frac{1}{2}\sqrt{2},\frac{1}{2}\sqrt{2}\right)$, which is a maximum.

9. Let $\gamma:[0,2\pi]\to\mathbb{R}^3$ be defined by $\gamma(t)=[4\cos(t),4\sin(t),-2\cos(t)]$. Calculate the curvature at the point $\left(-2\sqrt{2},2\sqrt{2},\sqrt{2}\right)$.

Solution We have $\gamma'(t)=[-4\sin(t),4\cos(t),2\sin(t)]$ and $\gamma''(t)=[-4\cos(t),-4\sin(t),2\cos(t)]$. The point in question corresponds to $t=3\pi/4$. We have $\gamma'(3\pi/4)=\left[-2\sqrt{2},-2\sqrt{2},\sqrt{2}\right]$ and $\gamma''(3\pi/4)=\left[2\sqrt{2},-2\sqrt{2},-\sqrt{2}\right]$. Therefore

$$\gamma'(3\pi/4)\times\gamma''(3\pi/4)=\det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2\sqrt{2} & -2\sqrt{2} & \sqrt{2} \\ 2\sqrt{2} & -2\sqrt{2} & -\sqrt{2} \end{pmatrix}=8\mathbf{i}+16\mathbf{k}=\begin{pmatrix} 8 \\ 0 \\ 16 \end{pmatrix}$$

and $\|\gamma'(3\pi/4)\times\gamma''(3\pi/4)\|=\sqrt{8^2+16^2}=8\sqrt{5}$. Since

$$\|\gamma'(3\pi/4)\|=\left\|\left[-2\sqrt{2},-2\sqrt{2},\sqrt{2}\right]\right\|=\sqrt{\left(-2\sqrt{2}\right)^2+\left(-2\sqrt{2}\right)^2+\left(\sqrt{2}\right)^2}=3\sqrt{2}$$

we have

$$\kappa=\frac{8\sqrt{5}}{\left(3\sqrt{2}\right)^3}=\frac{2}{27}\sqrt{10}.$$