Math 318 Fall 2008
Exam 3

1. Let S be the 2-manifold in $\mathbb{R}^3$ parameterized by

$$\gamma\left(\left[\begin{array}{c} r \\ \theta \end{array}\right]\right)=\left[\begin{array}{c} r\cos(\theta) \\ r\sin(\theta) \\ r^2 \end{array}\right], \qquad \left(1<r<4, 0<\theta<\frac{\pi}{2}\right).$$

Calculate the surface area of S.

Solution Since $D(\gamma)\left(\left[\begin{array}{c} r \\ \theta \end{array}\right]\right)=\left[\begin{array}{cc} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \\ 2r & 0 \end{array}\right]$, we have

$${}^tD(\gamma)\left(\left[\begin{array}{c} r \\ \theta \end{array}\right]\right)D(\gamma)\left(\left[\begin{array}{c} r \\ \theta \end{array}\right]\right)=\left[\begin{array}{cc} \cos^2(\theta)+\sin^2(\theta)+4r^2 & 0 \\ 0 & r^2\sin^2(\theta)+r^2\cos^2(\theta) \end{array}\right]=\left[\begin{array}{cc} 1+4r^2 & 0 \\ 0 & r^2 \end{array}\right]$$

and

$$\sqrt{\det\left({}^tD(\gamma)\left(\left[\begin{array}{c} r \\ \theta \end{array}\right]\right)D(\gamma)\left(\left[\begin{array}{c} r \\ \theta \end{array}\right]\right)\right)}=r\sqrt{1+4r^2}.$$

The surface area is therefore

$$\int_1^4\int_0^{\pi/2} r\sqrt{1+4r^2}\,drd\theta=\frac{5\pi}{24}\left(13\sqrt{65}-\sqrt{5}\right).$$

2. Calculate $\iint_Q \left(2x^2+y\right)\,dxdy$ where Q is the region between the hyperbolas $xy=1$, $xy=2$ and the parabolas $y=x^2$, $y=x^2+1$.

Solution Let $f\left(\left[\begin{array}{c} u \\ v \end{array}\right]\right)=1$.The map $T:Q\to\mathbb{R}^2$ defined by $\left[\begin{array}{c} u \\ v \end{array}\right]=T\left(\left[\begin{array}{c} x \\ y \end{array}\right]\right)=\left[\begin{array}{c} xy \\ y-x^2 \end{array}\right]$ has nonvanishing Jacobian determinant

$$J_T\left(\left[\begin{array}{c} x \\ y \end{array}\right]\right)=\det\left(\left[\begin{array}{cc} y & x \\ -2x & 1 \end{array}\right]\right)=2x^2+y$$

on Q and image $R=T(Q)=\left\{\left[\begin{array}{c} u \\ v \end{array}\right]\in\mathbb{R}^2 : 1<u<2, 0<v<1\right\}$. Therefore,

$$\iint_Q \left(2x^2+y\right)\,dxdy=\iint_Q f\left(T\left(\left[\begin{array}{c} x \\ y \end{array}\right]\right)\right)\cdot\left|J_T\left(\left[\begin{array}{c} x \\ y \end{array}\right]\right)\right|\,dxdy=\iint_R f\left(\left[\begin{array}{c} u \\ v \end{array}\right]\right)\,dudv=\iint_R 1\,dudv=1.$$

3. Let $U=\left\{\left[\begin{array}{c} s \\ t \end{array}\right]\in\mathbb{R}^2 : 0<s<t<1\right\}$ and $\gamma\left(\left[\begin{array}{c} s \\ t \end{array}\right]\right)=\left[\begin{array}{c} st \\ s+t \\ 2s+t \\ s^2+t^2 \end{array}\right]$ for $\left[\begin{array}{c} s \\ t \end{array}\right]\in U$. Calculate

$$\int_{\gamma(U)} \left(x_3-x_2\right)\,dx_1\wedge dx_4$$

Solution We have

$$\overrightarrow{\mathbf{u}}=D_1(\gamma)\left(\left[\begin{array}{c} s \\ t \end{array}\right]\right)=\left[\begin{array}{c} t \\ 1 \\ 2 \\ 2s \end{array}\right] \quad\text{and}\quad \overrightarrow{\mathbf{v}}=D_2(\gamma)\left(\left[\begin{array}{c} s \\ t \end{array}\right]\right)=\left[\begin{array}{c} s \\ 1 \\ 1 \\ 2t \end{array}\right].$$

Therefore,

$$dx_1 \wedge dx_4 \left(\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}\right) = 2\left(t^2 - s^2\right)$$

and

$$\int_{\gamma(U)} (x_3 - x_2)\, dx_1 \wedge dx_4 = \int_0^1 \int_0^{1-t} ((2s+t) - (s+t))\, 2\left(t^2 - s^2\right) dsdt = -\frac{1}{15}.$$

4. Let $M = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} : x^2 + y^2 + z^2 + w^2 < 1 \right\}$ and $\partial(M)$ have compatible orientations. Find a differential form ω such that

$$\int_M \omega = \int_{\partial(M)} \left(5x^2y^3z^2 + 7/w\right) dx \wedge dy \wedge dz + \left(5xyw + z^5\right) dx \wedge dy \wedge dw + \left(3xy^2 - z^3w^4\right) dx \wedge dz \wedge dw.$$

Solution By Stokes's Theorem, the exterior derivative

$$\left(7/w^2 + 5z^4 - 6xy\right) dx \wedge dy \wedge dz \wedge dw$$

of the form integrated over $\partial(M)$ will do.

5. Let $M = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x^2 + y^2 + z^2 < 1 \right\}$ and let $\partial(M)$ be the boundary of M with orientation induced by $\overrightarrow{\mathbf{n}}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ at each point $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \partial(M)$. Let $\mathbf{F}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x^2 \\ 1 \\ z \end{bmatrix}$. Calculate $\int_{\partial(M)} \Phi_{\mathbf{F}}$. (You may use whatever method you prefer.)

Solution Since $d\Phi_{\mathbf{F}} = M_{\text{div}(\mathbf{F})} = (2x+1)\, dx \wedge dy \wedge dz$, we have

$$\int_{\partial(M)} \Phi_{\mathbf{F}} = \int_M (2x+1)\, dx \wedge dy \wedge dz = 2\int_M x\, dx \wedge dy \wedge dz + \int_M dx \wedge dy \wedge dz = 0 + \frac{4}{3}\pi \cdot 1^3 = \frac{4}{3}\pi.$$

The direct calculation without Stokes Theorem uses the parameterization

$$\gamma\left(\begin{bmatrix} \phi \\ \theta \end{bmatrix}\right) = \begin{bmatrix} \cos(\theta)\sin(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\phi) \end{bmatrix}, \quad 0 \le \phi < \pi, 0 \le \theta < 2\pi.$$

Then we have

$$\overrightarrow{\mathbf{u}} = D_1(\gamma)\left(\begin{bmatrix} \phi \\ \theta \end{bmatrix}\right) = \begin{bmatrix} \cos(\theta)\cos(\phi) \\ \sin(\theta)\cos(\phi) \\ -\sin(\phi) \end{bmatrix} \quad \text{and} \quad \overrightarrow{\mathbf{v}} = D_2(\gamma)\left(\begin{bmatrix} \phi \\ \theta \end{bmatrix}\right) = \begin{bmatrix} -\sin(\theta)\sin(\phi) \\ \cos(\theta)\sin(\phi) \\ 0 \end{bmatrix}$$

and

$$dy \wedge dz\left(\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}\right) = -\cos(\theta)\sin^2(\phi), \; dx \wedge dz\left(\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}\right) = -\sin(\theta)\sin^2(\phi), \text{ and } dx \wedge dy\left(\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}\right) = \cos(\phi)\sin(\phi).$$

Also, $\mathbf{F}\left(\gamma\left(\begin{bmatrix} \phi \\ \theta \end{bmatrix}\right)\right) = \begin{bmatrix} \cos^2(\theta)\sin^2(\phi) \\ 1 \\ \cos(\theta)\sin(\phi) \end{bmatrix}$. Therefore,

$$\mathbf{F}_1\left(\gamma\left(\begin{bmatrix} \phi \\ \theta \end{bmatrix}\right)\right) dy \wedge dz\left(\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}\right) - \mathbf{F}_2\left(\gamma\left(\begin{bmatrix} \phi \\ \theta \end{bmatrix}\right)\right) dx \wedge dz\left(\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}\right) + \mathbf{F}_3\left(\gamma\left(\begin{bmatrix} \phi \\ \theta \end{bmatrix}\right)\right) dx \wedge dy\left(\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}\right)$$

$$= \cos^2(\theta)\sin^2(\phi)\cdot\left(-\cos(\theta)\sin^2(\phi)\right) - 1\cdot\left(-\sin(\theta)\sin^2(\phi)\right) + \cos(\phi)\cdot(\cos(\phi)\sin(\phi))$$
$$= -\cos^3(\theta)\sin^4(\phi) + \sin(\theta)\sin^2(\phi) + \cos^2(\phi)\sin(\phi)$$

and

$$\int_{\partial(M)} \Phi_{\mathbf{F}} = \int_0^{2\pi}\int_0^{\pi} \left(-\cos^3(\theta)\sin^4(\phi) + \sin(\theta)\sin^2(\phi) + \cos^2(\phi)\sin(\phi)\right) d\phi d\theta$$
$$= 0 + 0 + \frac{4}{3}\pi = \frac{4}{3}\pi\,.$$

6. Let $M = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x^2 + z^2 = 1, 0 < x < 1, 0 < z < 1, 0 < y < 2 \right\}$ be oriented by $\overrightarrow{\mathbf{n}}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \\ z \end{bmatrix}$ at each point $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in M$. Let $\partial(M)$ be the boundary of M with induced orientation. Let $\mathbf{F}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ x^2 - 1 \\ z \end{bmatrix}$. Calculate $\int_{\partial(M)} W_{\mathbf{F}}$ directly and by using Stokes's Theorem.

Solution Let $I = [0, 2]$ and $J = [0, \pi/2]$. Define $\gamma_1 : I \to \mathbb{R}^3$, $\gamma_2 : J \to \mathbb{R}^3$, $\gamma_3 : I \to \mathbb{R}^3$, and $\gamma_4 : J \to \mathbb{R}^3$ by

$$\gamma_1(t) = \begin{bmatrix} 1 \\ t \\ 0 \end{bmatrix}, \quad \gamma_2(t) = \begin{bmatrix} \cos(t) \\ 2 \\ \sin(t) \end{bmatrix}, \quad \gamma_3(t) = \begin{bmatrix} 0 \\ 2-t \\ 1 \end{bmatrix}, \quad \text{and} \quad \gamma_4(t) = \begin{bmatrix} \sin(t) \\ 0 \\ \cos(t) \end{bmatrix}.$$

These four functions parameterize the four pieces of $\partial(M)$, preserving the stated orientation. Since

$$\gamma_1'(t) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \gamma_2'(t) = \begin{bmatrix} -\sin(t) \\ 0 \\ \cos(t) \end{bmatrix}, \quad \gamma_3'(t) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \gamma_4'(t) = \begin{bmatrix} \cos(t) \\ 0 \\ -\sin(t) \end{bmatrix},$$

we have

$$\int_{\gamma_1(I)} W_{\mathbf{F}} = \int_0^2 \left(0\cdot dx\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) + \left(1^2 - 1\right)\cdot dy\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) + 0\cdot dz\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)\right) dt = 0,$$

$$\int_{\gamma_2(J)} W_{\mathbf{F}} = \int_0^{\pi/2} \left(0\cdot dx\left(\begin{bmatrix} -\sin(t) \\ 0 \\ \cos(t) \end{bmatrix}\right) + \left(\cos^2(t) - 1\right)\cdot dy\left(\begin{bmatrix} -\sin(t) \\ 0 \\ \cos(t) \end{bmatrix}\right) + \sin(t)\cdot dz\left(\begin{bmatrix} -\sin(t) \\ 0 \\ \cos(t) \end{bmatrix}\right)\right)$$

or

$$\int_{\gamma_2(J)} W_{\mathbf{F}} = \int_0^{\pi/2} \left(0\cdot(-\sin(t)) + \left(\cos^2(t) - 1\right)\cdot(0) + \sin(t)\cdot\cos(t)\right) dt = \frac{1}{2},$$

$$\int_{\gamma_3(J)} W_{\mathbf{F}} = \int_0^1 \left(0\cdot dx\left(\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}\right) + (0-1)\cdot dy\left(\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}\right) + 1\cdot dz\left(\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}\right)\right) dt$$

or

$$\int_{\gamma_3(J)} W_{\mathbf{F}} = \int_0^2 \left(0\cdot 0 + (0-1)\cdot(-1) + 1\cdot(0)\right) dt = 2,$$

and

$$\int_{\gamma_4(J)} W_{\mathbf{F}} = \int_0^{\pi/2} \left(0\cdot dx\left(\begin{bmatrix} \cos(t) \\ 0 \\ -\sin(t) \end{bmatrix}\right) + \left(\sin^2(t) - 1\right)\cdot dy\left(\begin{bmatrix} \cos(t) \\ 0 \\ -\sin(t) \end{bmatrix}\right) + \cos(t)\cdot dz\left(\begin{bmatrix} \cos(t) \\ 0 \\ -\sin(t) \end{bmatrix}\right)\right)$$

or

$$\int_{\gamma_4(J)} W_{\mathbf{F}} = \int_0^{\pi/2} \left(0 \cdot \cos(t) + \left(\sin^2(t) - 1\right) \cdot 0 + \cos(t) \cdot (-\sin(t))\right) dt = -\frac{1}{2}.$$

Therefore,

$$\int_{\partial(M)} W_{\mathbf{F}} = 0 + \frac{1}{2} + 2 + \left(-\frac{1}{2}\right) = 2.$$

Next, we calculate

$$\mathbf{curl}(\mathbf{F})\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 2x \end{bmatrix}.$$

If we parameterize M by $\gamma : I \times J \to \mathbb{R}^3$ with

$$\gamma\left(\begin{bmatrix} s \\ t \end{bmatrix}\right) = \begin{bmatrix} \cos(t) \\ s \\ \sin(t) \end{bmatrix},$$

then we have

$$\mathbf{curl}(\mathbf{F})\left(\gamma\left(\begin{bmatrix} s \\ t \end{bmatrix}\right)\right) = \begin{bmatrix} 0 \\ 0 \\ 2\cos(t) \end{bmatrix},$$

$$\overrightarrow{\mathbf{u}} = D_1(\gamma)\left(\begin{bmatrix} s \\ t \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \overrightarrow{\mathbf{v}} = D_2(\gamma)\left(\begin{bmatrix} s \\ t \end{bmatrix}\right) = \begin{bmatrix} -\sin(t) \\ 0 \\ \cos(t) \end{bmatrix}$$

and

$$dy \wedge dz\left(\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}\right) = \cos(t), \quad dx \wedge dz\left(\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}\right) = 0, \quad \text{and} \quad dx \wedge dy\left(\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}\right) = \sin(t).$$

Therefore

$$\int_M \Phi_{\nabla\times\mathbf{F}} = \iint_{I\times J} \left(0 dy \wedge dz\left(\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}\right) - 0 dx \wedge dz\left(\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}\right) + 2\cos(t)\, dx \wedge dy\left(\overrightarrow{\mathbf{u}}, \overrightarrow{\mathbf{v}}\right)\right) dsdt$$

or

$$\int_M \Phi_{\nabla\times\mathbf{F}} = \int_0^{\pi/2} \int_0^2 \left(2\cos(t)\sin(t)\right)\, dsdt = 2.$$