Math 318 Fall 2008
Exam 1-VP Debate Edition*
*Choose one question to avoid.

1. Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^3$ is linear, that $T(\mathbf{e}_1 + \mathbf{e}_2) = 5\mathbf{e}_1 + 3\mathbf{e}_2 + 4\mathbf{e}_3$, that $T(\mathbf{e}_1 - \mathbf{e}_2) = \mathbf{e}_1 - \mathbf{e}_2 - 4\mathbf{e}_3$, and that $T(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = 10(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$. What is the matrix representation $[T]$ of T with respect to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbb{R}^3$.

Solution Since

$$T(\mathbf{e}_1) = T\left(\frac{1}{2}((\mathbf{e}_1 + \mathbf{e}_2) + (\mathbf{e}_1 - \mathbf{e}_2))\right) = \frac{1}{2}(T(\mathbf{e}_1 + \mathbf{e}_2) + T(\mathbf{e}_1 - \mathbf{e}_2)) = \frac{1}{2}\left(\begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ -4 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

and

$$T(\mathbf{e}_2) = T(\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_1) = T(\mathbf{e}_1 + \mathbf{e}_2) - T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

and

$$T(\mathbf{e}_3) = M(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 - \mathbf{e}_1 - \mathbf{e}_2) = T(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) - T(\mathbf{e}_1) - T(\mathbf{e}_2) = \begin{bmatrix} 10 \\ 10 \\ 10 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 6 \end{bmatrix}.$$

Therefore,

$$[T] = \begin{bmatrix} 3 & 2 & 5 \\ 1 & 2 & 7 \\ 0 & 4 & 6 \end{bmatrix}.$$

2. By using the definition of an open set and being explicit about radii, show that the square $S = \left\{\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : 0 < x < 1, 0 < y < 1\right\}$ is open.

Solution Let $p = \begin{bmatrix} a \\ b \end{bmatrix} \in S$. If $r = \min(a, 1-a, b, 1-b)$ then $r > 0$ and $B_r(p) \subset S$. In other words, every point of S is the center of a disc that is entirely contained in S and that means that S is open.

3. Suppose that $f : \mathbb{R}^3 \to \mathbb{R}^2$, the point $\mathbf{a} \in \mathbb{R}^3$, and the vector $\mathbf{v} \in \mathbb{R}^3$ are defined by

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2y - z^2 \\ 3x^2y + z \end{bmatrix}, \qquad \mathbf{a} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ -5 \end{bmatrix}.$$

Calculate the Jacobian matrix $[J(f)(\mathbf{a})]$ and the directional derivative $D_{\mathbf{v}}(f)(\mathbf{a})$.

Solution We have

$$J(f)\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} \frac{\partial}{\partial x}(2y - z^2) & \frac{\partial}{\partial y}(2y - z^2) & \frac{\partial}{\partial z}(2y - z^2) \\ \frac{\partial}{\partial x}(3x^2y + z) & \frac{\partial}{\partial y}(3x^2y + z) & \frac{\partial}{\partial z}(3x^2y + z) \end{bmatrix} = \begin{bmatrix} 0 & 2 & -2z \\ 6xy & 3x^2 & 1 \end{bmatrix}.$$

Therefore,

$$J(f)\left(\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 0 & 2 & -4 \\ 18 & 3 & 1 \end{bmatrix} \quad \text{and} \quad D_{\mathbf{v}}(f)(\mathbf{a}) = \begin{bmatrix} 0 & 2 & -4 \\ 18 & 3 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ -5 \end{bmatrix} = \begin{bmatrix} 18 \\ 28 \end{bmatrix}.$$

4. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$, the point $\mathbf{a} \in \mathbb{R}^2$, and the input increment $\mathbf{h} \in \mathbb{R}^2$ are defined by

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} xy \\ x^2 - y \end{bmatrix}, \qquad \mathbf{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{h} = \begin{bmatrix} 0.1 \\ -0.2 \end{bmatrix}.$$

What is the magnitude of the (vector) error that results when the actual increment $f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a})$ is estimated by its best linear approximation at $\mathbf{a}$?

Solution We calculate

$$D(f)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y & x \\ 2x & -1 \end{bmatrix} \qquad \text{and} \qquad D(f)(\mathbf{a}) = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$$

The magnitude of the error is

$$\left\| f\left(\begin{bmatrix} 1.1 \\ -0.2 \end{bmatrix}\right) - f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) - D(f)(\mathbf{a})(\mathbf{h}) \right\| = \left\| \begin{bmatrix} (1.1)(-0.2) \\ (1.1)^2 - (-0.2) \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0.1 \\ -0.2 \end{bmatrix} \right\|.$$

Thus, the magnitude of the error evaluates to

$$\left\| \begin{bmatrix} -0.22 \\ 1.41 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -0.2 \\ 0.4 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -0.02 \\ 0.01 \end{bmatrix} \right\| = \sqrt{(-0.02)^2 + (0.01)^2} = 0.02236\ldots.$$

5. Let $\varphi = f \circ g$ where $f : \mathbb{R}^3 \to \mathbb{R}^2$ is the function of the third problem and $g : \mathbb{R} \to \mathbb{R}^3$ is defined by

$$g(u) = \begin{bmatrix} u \\ 3u \\ u^3 + 1 \end{bmatrix}.$$

Use the Chain Rule to calculate $D(\varphi)(1)$.

Solution Notice that

$$g(1) = \mathbf{a} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}.$$

We calculate

$$D(f)(\mathbf{a}) = \begin{bmatrix} 0 & 2 & -4 \\ 18 & 3 & 1 \end{bmatrix} \text{ (from question 3)}, \qquad D(g)(u) = \begin{bmatrix} 1 \\ 3 \\ 3u^2 \end{bmatrix}, \qquad \text{and} \qquad D(g)(u) = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}.$$

Therefore,

$$D(\varphi)(1) = \begin{bmatrix} 0 & 2 & -4 \\ 18 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -6 \\ 30 \end{bmatrix}.$$

We can verify this by an unpleasant direct calculation:

$$\varphi(u) = \begin{bmatrix} 2(3u) - (u^3 + 1)^2 \\ 3u^2(3u) + (u^3 + 1) \end{bmatrix} = \begin{bmatrix} -u^6 - 2u^3 + 6u - 1 \\ 10u^3 + 1 \end{bmatrix}, \quad D(\varphi)(u) = \begin{bmatrix} -6u^5 - 6u^2 + 6 \\ 30u^2 \end{bmatrix},$$

and

$$D(\varphi)(1) = \begin{bmatrix} -6 \\ 30 \end{bmatrix}.$$

6. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 + y \\ xy \end{bmatrix}$. At which points does the Inverse Function Theorem guarantee that f is locally invertible? The point $\mathbf{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is one of these. If g is the local inverse of f defined on a neighborhood of $\mathbf{b} = f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$, then what is $D(g)(\mathbf{b})$?

Solution Since

$$\det\left(D(f)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)\right) = \det\left(\begin{bmatrix} 2x & 1 \\ y & x \end{bmatrix}\right) = 2x^2 - y,$$

the Inverse Function Theorem guarantees that f is locally invertible at every point $\begin{bmatrix} x \\ y \end{bmatrix}$ that is not on the parabola $y = 2x^2$. We have

$$D(g)(\mathbf{b}) = D(f)(\mathbf{a})^{-1} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ 0 & 1 \end{bmatrix}.$$

7. Notice that the point $\mathbf{a} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ lies on the curve $\mathcal{C}$ of intersection of the surfaces $x + z - 1 = 0$ and $x + y^3 - 3yz - 2 = 0$. Show that z can be used to parameterize $\mathcal{C}$ sufficiently close to $\mathbf{a}$. What about y?

Solution Let $F\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + z - 1 \\ x + y^3 - 3yz - 2 \end{bmatrix}$. Then

$$D(F)\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 3y^2 - 3z & -3y \end{bmatrix}.$$

The map

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 1 \\ 1 & 3\left(y^2 - z\right) & -3y \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 3\left(y^2 - z\right) \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

is invertible at all points for which $z \neq y^2$. The point $\mathbf{a}$ satisfies $F(\mathbf{a}) = \mathbf{0}$ and its third entry is not the square of its second entry. Therefore, the Implicit Function Theorem states that there is an $r > 0$ and a function φ from $(-r, r)$ to a neighborhood of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ in $\mathbb{R}^2$ such that $\varphi(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $F\left(\begin{bmatrix} \varphi(z) \\ z \end{bmatrix}\right) = \mathbf{0}$. In other words, $z \mapsto \begin{bmatrix} \varphi_1(z) \\ \varphi_2(z) \\ z \end{bmatrix}$ for $|z| < r$ is a parameterization of the curve of intersection near $\mathbf{a}$. Similarly, the map

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 1 \\ 1 & 3\left(y^2 - z\right) & -3y \end{bmatrix}\begin{bmatrix} u_1 \\ 0 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -3y \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

is invertible at all points for which $y \neq -1/3$. The point $\mathbf{a}$ satisfies $F(\mathbf{a}) = \mathbf{0}$ and its second entry is not $-1/3$. Therefore, the Implicit Function Theorem states that there is an $\rho > 0$ and a function ψ from $(1 - \rho, 1 + \rho)$ to a neighborhood of $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ in $\mathbb{R}^2$ such that $\psi(1) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ and $F\begin{pmatrix} \psi_1(y) \\ y \\ \psi_2(y) \end{pmatrix} = \mathbf{0}$. In other words, $y \mapsto \begin{bmatrix} \psi_1(y) \\ y \\ \psi_2(y) \end{bmatrix}$ for $|z| < \rho$ is a parameterization of the curve of intersection near $\mathbf{a}$.