# Math 318 Fall 2008
# Exam 2

1. Let $\gamma : \mathbb{R}^1 \to \mathbb{R}^3$ be defined by
$$\gamma(t) = \begin{bmatrix} t+1 \\ t-1 \\ t^2-1 \end{bmatrix}.$$
Give two different reasons—one using $\gamma$ and one using another function of your devising—that explain why $M = \left\{\gamma(t) : t \in \mathbb{R}^1\right\}$ is a smooth 1-dimensional manifold embedded in $\mathbb{R}^3$. Each explanation should use involve the derivative of the defining function. (These derivatives will be used in the next problem.)

**Solution** Observe that $D(\gamma)(t)$ exists for all $t$ and, for $v \in \mathbb{R}^1$, we have
$$D(\gamma)(t)(v) = \begin{bmatrix} 1 \\ 1 \\ 2t \end{bmatrix}[v] = \begin{bmatrix} v \\ v \\ 2tv \end{bmatrix},$$
which is $\overrightarrow{\mathbf{0}}$ if and only if $v = 0$. Therefore, $D(\gamma)(t) : \mathbb{R} \to \mathbb{R}^3$ is 1-1 for all $t$. Therefore, $M$ is a smooth 1-manifold in $\mathbb{R}^3$. Alternatively, we can let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by
$$F\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x-y-2 \\ xy-z \end{bmatrix}.$$
Notice that
$$M = \left\{\gamma(t) : t \in \mathbb{R}^1\right\} = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : F\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}.$$
Observe that $D(F)\left(\overrightarrow{\mathbf{x}}\right) : \mathbb{R}^3 \to \mathbb{R}^2$ is given by
$$D(F)\left(\overrightarrow{\mathbf{x}}\right)\left[\overrightarrow{\mathbf{u}}\right] = \begin{bmatrix} 1 & -1 & 0 \\ y & x & -1 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u-v \\ yu+xv-w \end{bmatrix} \text{ where } x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } \overrightarrow{\mathbf{u}} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}.$$
The first and third columns of $D(F)\left(\overrightarrow{\mathbf{x}}\right)$ are clearly linearly independent, so the image of $D(F)\left(\overrightarrow{\mathbf{x}}\right)$ is at least two dimensional (and of course cannot have a greater dimension). Thus $D(F)\left(\overrightarrow{\mathbf{x}}\right)$ is onto for $\overrightarrow{\mathbf{x}} \in M$. This also shows that $M$ is a smooth manifold. Another possibility is to let $x = t+1$. Then $t-1 = x-2$ and $t^2-1 = (t+1)(t-1) = x(x-2)$. The graph $\Gamma(f)$ of the differentiable function $f : \mathbb{R}^1 \to \mathbb{R}^2$ defined by
$$f(x) = \begin{bmatrix} x-2 \\ x(x-2) \end{bmatrix}$$
is
$$M = \Gamma(f) = \left\{\begin{bmatrix} x \\ x-2 \\ x(x-2) \end{bmatrix} \in \mathbb{R}^3 : x \in \mathbb{R}^1\right\}.$$
There is no need to investigate injectivity or surjectivity of $D(f)(x)$ in this presentation of $M$.

2. Let $M$ be the manifold of the preceding problem. Use each representation of $M$ in the preceding exercise to calculate the tangent space $T_{\overrightarrow{\mathbf{c}}}M$ where $\overrightarrow{\mathbf{c}} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$.

**Solution** Using $F$, we have $D(F)\left(\overrightarrow{\mathbf{c}}\right)\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 3 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u-v \\ u+3v-w \end{bmatrix}$. Since $T_{\overrightarrow{\mathbf{c}}}M = \ker\left(D(F)\left(\overrightarrow{\mathbf{c}}\right)\right)$, we have

$$T_{\overrightarrow{\mathbf{c}}}M = \left\{\begin{bmatrix} u \\ v \\ w \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} u-v \\ u+3v-w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\} = \left\{\begin{bmatrix} u \\ u \\ 4u \end{bmatrix} : u \in \mathbb{R}\right\}.$$

Using $\gamma$ and noting that $\overrightarrow{\mathbf{c}}$ corresponds to $t = 2$,we have

$$D(\gamma)(2)(v) = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}[v] = \begin{bmatrix} v \\ v \\ 4v \end{bmatrix}.$$

Thus, $T_{\overrightarrow{\mathbf{c}}}M = D(\gamma)\left(\overrightarrow{\mathbf{c}}\right)\left(\mathbb{R}^1\right) = \text{Image}(\gamma)$ is given by

$$T_{\overrightarrow{\mathbf{c}}}M = \left\{\begin{bmatrix} v \\ v \\ 4v \end{bmatrix} \in \mathbb{R}^3 : v \in \mathbb{R}\right\}.$$

Finally, using $f$, we note that $\overrightarrow{\mathbf{c}}$ corresponds to $x = 3$. We have $D(f)(x) = \begin{bmatrix} 1 \\ 2x-2 \end{bmatrix}$ and

$$D(f)(3)(v) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}[v] = \begin{bmatrix} v \\ 4v \end{bmatrix}.$$

Therefore,

$$T_{\overrightarrow{\mathbf{c}}}M = \Gamma(D(f)(3)) = \left\{\begin{bmatrix} v \\ v \\ 4v \end{bmatrix} \in \mathbb{R}^3 : v \in \mathbb{R}^1\right\}.$$

3. Calculate the degree 2 Taylor polynomial of $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x\sqrt{y}$ with base point $\overrightarrow{\mathbf{c}} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$.

**Solution** We calculate $D_{(1,0)}(f)(x,y) = \sqrt{y}$, $D_{(0,1)}(f)(x,y) = \frac{1}{2}xy^{-1/2}$, $D_{(2,0)}(f)(x,y) = 0$, $D_{(1,1)}(f)(x,y) = \frac{1}{2}y^{-1/2}$, and $D_{(0,2)}(f)(x,y) = -\frac{1}{4}xy^{-3/2}$. Therefore, $f(-1,4) = -2$, $D_{(1,0)}(f)(-1,4) = 2$, $D_{(0,1)}(f)(-1,4) = -1/4$, $D_{(2,0)}(f)(-1,4)/(2,0)! = 0$, $D_{(1,1)}(f)(-1,4)/(1,1)! = 1/4$, and $D_{(0,2)}(f)(-1,4)/(0,2)! = 1/64$. The required polynomial is

$$P_{f,\overrightarrow{\mathbf{c}},2}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = -2 + 2(x+1) - \frac{1}{4}(y-4) + \frac{1}{4}(x+1)(y-4) + \frac{1}{64}(y-4)^2.$$

4. Let $F(x,y) = (4+y)x + y^3 + y$. The equation $F(x,y) = 0$ defines $y$ as an implicit function $f$ of $x$ in a neighborhood of the origin. That is, there is a function $f$ such that $f(0) = 0$ and $(4+f(x))x + f(x)^3 + f(x) = 0$ for all $x$ sufficiently close to 0. What is the degree 3 Taylor polynomial of $f$ with base point 0?

**Solution** We have

$$D(f)(x) = -\frac{D_1(F)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)}{D_2(F)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)} = -\frac{4+y}{x+3y^2+1} \quad (y = f(x))$$

so setting $x = 0$ gives $D(f)(0) = -4$. The required Taylor polynomial is of the form $0 - 4x + ax^2 + bx^3$. Substituting $-4x + ax^2 + bx^3$ for $y$ in the equation $F(x,y) = 0$ and discarding terms of degree 4 or higher, we obtain

$$\left(4 - 4x + ax^2 + bx^3\right)x + \left(-4x + ax^2 + bx^3\right)^3 + \left(-4x + ax^2 + bx^3\right) = 0$$

or

$$\left(4x-4x^2+ax^3\right)+\left(-64x^3\right)+\left(-4x+ax^2+bx^3\right)=0$$

or

$$(a-4)\,x^2+(a-64+b)\,x^3=0.$$

We see that $a=4$ and $b=60$. The required polynomial is $-4x+4x^2+60x^3$.

5. What is the signature of $Q\,(x,y,z)=x^2+y^2-2xy-4xz+4yz$?

**Solution** We calculate

$$\begin{aligned} Q\,(x,y,z) &= x^2+y^2-2xy-4xz+4yz \\ &= x^2-2\,(y+2z)\,x+y^2+4yz \\ &= x^2-2\,(y+2z)\,x+(y+2z)^2+y^2+4yz-(y+2z)^2 \\ &= (x-y-2z)^2+y^2+4yz-(y+2z)^2 \\ &= (x-y-2z)^2-(2z)^2\,. \end{aligned}$$

$$(x-y-2z)^2-(2z)^2$$

If $A\,(x-y-2z)+B\,(2z)=0$ for all $x$, $y$, and $z$, then for $x=2$, $y=1$, $z=0$ we obtain $A=0$. It follows that $B\,(2z)=0$ for all $z$, which implies $B=0$. Thus, the linear functions

$$L_1\left(\begin{bmatrix} x\\ y\\ z\end{bmatrix}\right)=x-y-2z \text{ and } L_2\left(\begin{bmatrix} x\\ y\\ z\end{bmatrix}\right)=2z$$

are linearly independent. It follows that $Q$ has signature $(1,1)$.

6. Find and classify the critical points of $f\left(\begin{bmatrix} x\\ y\end{bmatrix}\right)=x^3+y^2-6xy$.

**Solution** We have

$$D_1\,(f)\left(\begin{bmatrix} x\\ y\end{bmatrix}\right)=3x^2-6y \qquad\text{and}\qquad D_2\,(f)\left(\begin{bmatrix} x\\ y\end{bmatrix}\right)=2y-6x.$$

The second equation tells us that $y=3x$ at a critical point. The first equation then tells us that $3x^2-6\,(3x)=0$ at a critical point. There are two solutions, $x=0$ and $x=6$. These solutions yield two critical points $P=\begin{bmatrix} 0\\ 0\end{bmatrix}$ and $Q=\begin{bmatrix} 6\\ 18\end{bmatrix}$. We calculate

$$D_{(2,0)}\,(f)\left(\begin{bmatrix} x\\ y\end{bmatrix}\right)=6x, \qquad D_{(1,1)}\,(f)\left(\begin{bmatrix} x\\ y\end{bmatrix}\right)=-6, \qquad D_{(0,2)}\,(f)\left(\begin{bmatrix} x\\ y\end{bmatrix}\right)=2.$$

Therefore

$$\operatorname{Discr}\left(f,\begin{bmatrix} x\\ y\end{bmatrix}\right)=(6x)\,(2)-(-6)^2=12\,(x-3)\,.$$

In particular, $\operatorname{Discr}(f,P)=-36<0$ and $\operatorname{Discr}(f,Q)=36>0$. We infer that there is a saddle point at $P$ and a local extremum at $Q$. Because $D_{(0,2)}\,(f)\,(Q)=2>0$, we deduce that the extremum at $Q$ is a local minimum.

7. Let

$$f\left(\begin{bmatrix} x\\ y\\ z\end{bmatrix}\right)=x^2+y^2+z^2/2, \qquad g\left(\begin{bmatrix} x\\ y\\ z\end{bmatrix}\right)=x+2y+z-70, \qquad h\left(\begin{bmatrix} x\\ y\\ z\end{bmatrix}\right)=x-z.$$

Minimize $f\left(\begin{bmatrix} x\\ y\\ z\end{bmatrix}\right)$ subject to the constraints $g\left(\begin{bmatrix} x\\ y\\ z\end{bmatrix}\right)=0$ and $h\left(\begin{bmatrix} x\\ y\\ z\end{bmatrix}\right)=0$.

**Solution** We set

$$\nabla(f)\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \lambda\nabla(g)\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) + \mu\nabla(h)\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right),$$

or $2x = \lambda + \mu$, $2y = 2\lambda$, $z = \lambda - \mu$. Since $x = z$ at a critical point, we have

$$\lambda + \mu = 2x = 2z = 2(\lambda - \mu) = 2\lambda - 2\mu$$

or

$$\lambda = 3\mu.$$

Our Lagrange multiplier equations become $x = 2\mu$, $y = 3\mu$, $z = 2\mu$. Substituting into the $g = 0$ constraint, we get $(2\mu) + 2(3\mu) + (2\mu) = 70$, or $\mu = 7$. It follows that $x = 2(7) = 14$, $y = 3(7) = 21$, $z = 2(7) = 17$. The minimum of $f$ is $14^2 + 21^2 + 14^2/2$, or 735.