Math 318 Fall 2008
Final **Exam**

1. Let $F : \mathbb{R}^4 \to \mathbb{R}^2$ be defined by

$$F(\mathbf{p}) = \begin{bmatrix} 2x + yz + w^2 \\ x^2 y + z \end{bmatrix}, \qquad \mathbf{p} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}.$$

From among the following list of expressions, calculate those that are defined:

$$D_2(F)(\mathbf{p}), \quad D(F)(\mathbf{p}), \quad D_3(F_2)(\mathbf{p}), \quad D(F_1)(\mathbf{p}), \quad \nabla(F)(\mathbf{p}), \quad \nabla(F_1)(\mathbf{p}).$$

2. Suppose that the function $f : \mathbb{R}^2 \to \mathbb{R}^3$, the point $\mathbf{a} \in \mathbb{R}^2$, and the vector $\mathbf{v} \in \mathbb{R}^2$ are defined by

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + y^3 \\ -x^2 \\ 4x - y^2 \end{bmatrix}, \qquad \mathbf{a} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

Calculate the directional derivative $D_{\mathbf{v}}(f)(\mathbf{a})$.

3. Suppose that the function $f : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$f\left(\begin{bmatrix} x \\ y \\ x \end{bmatrix}\right) = \begin{bmatrix} x + \exp(y) \\ y + \exp(z) \\ z + \exp(x) \end{bmatrix}.$$

Determine the points at which f is locally invertible.

4. Let $F : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$F\left(\begin{bmatrix} x \\ y \\ x \end{bmatrix}\right) = \begin{bmatrix} x + 2y + z \\ xz + 2y^2 + 24 \end{bmatrix}.$$

Let. $\mathcal{C} = \left\{(x, y, z) \in \mathbb{R}^3 : F(x, y, z) = 0, y > 0\right\}$. Use the Implicit Function Theorem to determine the points $\mathbf{c} \in \mathcal{C}$ at which there is an $r > 0$ such that $\mathcal{C} \cap B_r(\mathbf{c})$ can be parameterized by x.

5. The graph of $y = \exp(x)$, $0 < x < 1$ is rotated about the horizontal line $x = -2$. Show that the resulting surface of revolution is a smooth manifold.

6. Calculate the tangent space $T_{\mathbf{c}}(M)$ of

$$M = \left\{\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} : xy + z + 2w = 3, x + y + zw = 0\right\} \text{ at } \mathbf{c} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix}$$

7. Let $U = \left\{\begin{bmatrix} s \\ t \end{bmatrix} \in \mathbb{R}^2 : 0 < s < 1, 0 < t < 2\pi\right\}$. Let $\gamma : \mathbb{U} \to \mathbb{R}^3$ be defined by

$$\gamma\left(\begin{bmatrix} s \\ t \end{bmatrix}\right) = \begin{bmatrix} s\cos(t) \\ s\sin(t) \\ t \end{bmatrix}.$$

Calculate the surface area of $\gamma(U)$. You may use

$$2\int \sec^3(\xi)\, d\xi = \sec(\xi)\tan(\xi) + \ln(|\sec(\xi) + \tan(\xi)|) + C,$$

if needed.

8. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} u+v \\ u^2 - v \end{bmatrix}.$$

Let $U = \left\{\begin{bmatrix} u \\ v \end{bmatrix} \in \mathbb{R}^2 : 0 < u, 0 < v, u+v < 2\right\}$. Calculate

$$\iint_{T(U)} \frac{1}{\sqrt{1+4x+4y}} dydx.$$

9. Let $U = \left\{\begin{bmatrix} s \\ t \end{bmatrix} \in \mathbb{R}^2 : 0 < s < 1,\ 0 < t < 1\right\}$ and $\gamma\left(\begin{bmatrix} s \\ t \end{bmatrix}\right) = \begin{bmatrix} st \\ s+3t \\ 2s+t \\ s^2+t^2 \end{bmatrix}$ for $\begin{bmatrix} s \\ t \end{bmatrix} \in U$. Calculate

$$\int_{\gamma(U)} (2x_1 + x_4)\ dx_3 \wedge dx_2$$

10. Calculate the exterior derivative of

$$\omega = \left(5x^2 + z^2\right) dx \wedge dy + \left(5x^2 y + z^3\right) dy \wedge dz + \left(3xy^3 - z^2\right) dx \wedge dz.$$

11. Let $M = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : x^2+y^2+z^2 < 1, 0 < z\right\}$ and let $\partial(M)$ be the boundary of M with orientation induced by $\overrightarrow{\mathbf{n}}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ at each point $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \partial(M)$ with $0 < z$ and $\overrightarrow{\mathbf{n}}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$ at each point $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \in \partial(M)$. Let $\mathbf{F}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ z \end{bmatrix}$. Calculate $\int_{\partial(M)} \Phi_{\mathbf{F}}$ directly *and* by using Stokes's Theorem.

12. Let

$$M = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : x^2+y^2 = 1, 1 < z < 2\right\} \bigcup \left\{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} : x^2+y^2 < 1\right\}.$$

be oriented by $\overrightarrow{\mathbf{n}}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ at each point $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in M$ with $1 < z$ and $\overrightarrow{\mathbf{n}}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$ at each point $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \in M$. Let $\partial(M)$ be the boundary of M with induced orientation. Let

$$\mathbf{F}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} z \\ x \\ 2y \end{bmatrix}.$$

Calculate $\int_{\partial(M)} W_{\mathbf{F}}$ directly *and* by using Stokes's Theorem.