Chapter 8

Math 318 F09 HW 2

1.4.9 Compute the cross-products:

$$\begin{bmatrix} 2x \\ -y \\ 3z \end{bmatrix} \times \begin{bmatrix} x \\ 2y \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \times \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \qquad \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} \times \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$$

8.1 Solution:

$$\begin{bmatrix} 2x \\ -y \\ 3z \end{bmatrix} \times \begin{bmatrix} x \\ 2y \\ 0 \end{bmatrix} = \begin{bmatrix} (-y)\cdot(0) - (2y)(3z) \\ -((2x)\cdot(0) - (x)\cdot(3z)) \\ (2x)(2y) - (x)(-y) \end{bmatrix} = \begin{bmatrix} -6yz \\ 3xz \\ 5xy \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \times \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} (2)\cdot(3) - (0)(5) \\ -((1)\cdot(3) - (2)\cdot(5)) \\ (1)(0) - (2)(2) \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} \times \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} (-1)\cdot(2) - (0)(6) \\ -((2)\cdot(2) - (3)\cdot(6)) \\ (2)(0) - (3)(-1) \end{bmatrix} = \begin{bmatrix} -2 \\ 14 \\ 3 \end{bmatrix}$$

1.4.14 a,b Given that

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

a) Compute $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ and $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

b) Comfirm that $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) = 0$. What is the geometrical relationship between $\mathbf{v}$ and $\mathbf{v} \times \mathbf{w}$?

8.2 Solution

a)

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \times \left(\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 3 \end{bmatrix} \text{ and}$$

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = \left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right) \times \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}$$

b)

$$\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = (2)(0) + (0)(3) + (1)(0) = 0$$

The geometrical relationship between $\mathbf{v}$ and $\mathbf{v} \times \mathbf{w}$ is that they are perpendicular.

a) What is the area of the parallelogram with vertices at

$$P=\begin{pmatrix}0\\0\end{pmatrix},\quad Q=\begin{pmatrix}1\\2\end{pmatrix},\quad S=\begin{pmatrix}5\\1\end{pmatrix},\quad R=\begin{pmatrix}6\\3\end{pmatrix}?$$

b) What is the area of the parallelogram with vertices at

$$A=\begin{pmatrix}0\\0\end{pmatrix},\quad B=\begin{pmatrix}1\\2\end{pmatrix},\quad D=\begin{pmatrix}5\\-1\end{pmatrix},\quad C=\begin{pmatrix}6\\1\end{pmatrix}?$$

8.3 Solution

a) Observe that $\overrightarrow{PQ}=\overrightarrow{SR}=\begin{bmatrix}1\\2\end{bmatrix}$ and $\overrightarrow{QR}=\overrightarrow{PS}=\begin{bmatrix}5\\1\end{bmatrix}$. The area of paralellogram $PQRS$ is $\left|\det\left(\begin{bmatrix}1&5\\2&1\end{bmatrix}\right)\right|=9$.

b) Observe that $\overrightarrow{AB}=\overrightarrow{DC}=\begin{bmatrix}1\\2\end{bmatrix}$ and $\overrightarrow{BC}=\overrightarrow{AD}=\begin{bmatrix}5\\-1\end{bmatrix}$. The area of paralellogram $PQRS$ is $\left|\det\left(\begin{bmatrix}1&5\\2&-1\end{bmatrix}\right)\right|=11$.

1.5.2 Determine whether the given subset of the given superset is open, closed, or both, or neither, and say why.

a) xy-plane in $\mathbb{R}^3$
b) $\mathbb{R}\subset\mathbb{C}$
c) The line $x=5$ in the xy-plane
d) $(0,1)$ in $\mathbb{C}$
e) $\mathbb{R}^n$ in $\mathbb{R}^n$
f) The unit sphere in $\mathbb{R}^3$

8.4 Solution

a) xy-plane in $\mathbb{R}^3$

Closed because the complement $U=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}: z\neq 0\right\}$ is open. If $p=\begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix}\in U$, then $B_{|z_0|/2}(p)$ is a ball with positive radius contained in U.

Not open: In fact, the xy-plane does not contain a ball (in $\mathbb{R}^3$) about any of its points. If $p=\begin{pmatrix}x_0\\y_0\\0\end{pmatrix}$ is a point in the xy-plane (with any values x_0 and y_0 you care to use) and if ϵ is any positive number you care to choose, then $\begin{pmatrix}x_0\\y_0\\\epsilon/2\end{pmatrix}$ is in $B_\epsilon(p)$ but not in the xy-plane, so p is not the center of a ball contained in the xy-plane.

b) $\mathbb{R}\subset\mathbb{C}$

Closed because the complement $U=\left\{\begin{pmatrix}x\\y\end{pmatrix}: y\neq 0\right\}$ is open. If $p=\begin{pmatrix}x_0\\y_0\end{pmatrix}\in U$, then $B_{|y_0|/2}(p)$ is a disk with positive radius contained in U.

Not open: In fact, the $\mathbb{R}$ does not contain a disk (in $\mathbb{R}^2$) about any of its points. If $p = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}$ is a point in $\mathbb{R} \subset \mathbb{C}$ (with any value x_0 you care to use) and if ϵ is any positive number you care to choose, then $\begin{pmatrix} x_0 \\ \epsilon/2 \end{pmatrix}$ is in $B_\epsilon(p)$ but not in $\mathbb{R} \subset \mathbb{C}$, so p is not the center of a ball contained in $\mathbb{R}$.

c) The line $x = 5$ in the xy-plane

Closed because the complement $U = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x \neq 5 \right\}$ is open. If $p = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \in U$, then $B_{|x_0-5|/2}(p)$ is a disk with positive radius contained in U.

Not open: In fact, the $x = 5$ does not contain a disk (in $\mathbb{R}^2$) about any of its points. If $p = \begin{pmatrix} 5 \\ y_0 \end{pmatrix}$ is a point on the given line (with any value y_0 you care to use) and if ϵ is any positive number you care to choose, then $\begin{pmatrix} 5+\epsilon/2 \\ y_0 \end{pmatrix}$ is in $B_\epsilon(p)$ but not on the line $x = 5$, so p is not the center of a ball contained in the line.

d) $(0, 1)$ in $\mathbb{C}$

Not closed: The point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a boundary point of the interval $(0, 1)$, but it is not in the interval. Thus, $(0, 1)$ does not contain all its boundary points and is therefore not closed.

Not open: In fact, $(0, 1)$ does not contain a disk (in $\mathbb{R}^2$) about any of its points. If $p = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}$ is a point in $(0, 1)$ (with any value x_0 between 0 and 1 you care to use) and if ϵ is any positive number you care to choose, then $\begin{pmatrix} x_0 \\ \epsilon/2 \end{pmatrix}$ is in $B_\epsilon(p)$ but not in $(0, 1)$, so p is not the center of a ball contained in $(0, 1)$.

e) $\mathbb{R}^n$ in $\mathbb{R}^n$

Both open and closed. If $p \in \mathbb{R}^n$, then p is the center of $B_1(p)$ (or any pother positive radius you care to choose) and $B_1(p) \subset \mathbb{R}^n$. Therefore, $\mathbb{R}^n$ is an open subset of $\mathbb{R}^n$. $\mathbb{R}^n$ is also closed, and that is because its complement, the empty set $\emptyset$, is open. (If want to contradict this statement, that is, if you want to show that $\emptyset$ is *not* open, then you have to find a point $p \in \emptyset$ for which $B_\epsilon(p)$ is not contained in $\emptyset$ for every positive ϵ. Good luck finding such a point in the set with no points.)

f) The unit sphere in $\mathbb{R}^3$

Closed: The complement $U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x^2 + y^2 + x^2 \neq 1 \right\}$ is open. If $p \in U$, then $\rho = \frac{1}{2} \left| \|\mathbf{p}\| - 1 \right| > 0$. If $q \in B_\rho(p)$, then $\|\mathbf{q}\| > 1$ if $\|\mathbf{p}\| > 1$ and $\|\mathbf{q}\| < 1$ if $\|\mathbf{p}\| < 1$. (Draw a picture!) Therefore $B_\rho(p) \subset U$ and U is open.

Not open. The point $p = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ belongs to the unit sphere. If $\epsilon > 0$, then $\begin{pmatrix} 1 \\ \epsilon/2 \\ 0 \end{pmatrix}$ belongs to $B_\epsilon(p)$ but not to the unit sphere, so the unit sphere does not contain any open ball centered at p (or, in fact, at any of its points).

1.5.5 Determine whether the given subset of $\mathbb{R}$ or $\mathbb{R}^2$ is open, closed, both, or neither, and say why.

a) $S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : 1 < x^2 + y^2 < 2 \right\}$

b) $T = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : xy \neq 0 \right\}$

c) $X = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : y = 0 \right\}$

d) $\mathbb{Q} \subset \mathbb{R}$

8.5 Solution

a) $S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : 1 < x^2 + y^2 < 2 \right\}$

Open: Suppose that p_0 be longs to S. Let ρ be the distance of p_0 to the origin. Then $1 < \rho < \sqrt{2}$. Let $\epsilon = \frac{1}{2} \min \left(\rho - 1, \sqrt{2} - \rho \right)$. A sketch shows that $B_\epsilon (p_0) \subset S$.

Not closed: Let $q = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. If $\varepsilon > 0$, then $\begin{pmatrix} 1 - \varepsilon/2 \\ 0 \end{pmatrix} \in B_\varepsilon (q) \cap \complement S$ and $\begin{pmatrix} 1 + \varepsilon/2 \\ 0 \end{pmatrix} \in B_\varepsilon (q) \cap \complement S$. Thus, every open ball centered at q intersects both S and the complement of S. It follows that $q \in \partial (S) \setminus S$. In other words, S does not contain all its boundary points, so S is not closed.

b) $T = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : xy \neq 0 \right\} \mathbb{R} \subset \mathbb{C}$

Open

Not closed: In fact, the $\mathbb{R}$ does not contain a disk (in $\mathbb{R}^2$) about any of its points. If $p = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}$ is a point in $\mathbb{R} \subset \mathbb{C}$ (with any value x_0 you care to use) and if ϵ is any positive number you care to choose, then $\begin{pmatrix} x_0 \\ \epsilon/2 \end{pmatrix}$ is in $B_\epsilon (p)$ but not in $\mathbb{R} \subset \mathbb{C}$, so p is not the center of a ball contained in $\mathbb{R}$.

c) $X = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : y = 0 \right\}$

Closed and not open. This is the same as 1.5.2b.

d) $\mathbb{Q} \subset \mathbb{R}$

Not closed and not open: every open interval has both a rational number in it (so $\mathbb{R} \setminus \mathbb{Q}$ is not open, hence $\mathbb{Q}$ is not closed) and an irrational number in it (so $\mathbb{Q}$ is not open).